Flag-no-square triangulations and Gromov boundaries in dimension 3

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Abstract

We indicate an infinite family of 3–dimensional topological spaces, which are homeomorphic to boundaries of certain word–hyperbolic groups. The groups are right angled hyperbolic Coxeter groups, whose nerves are flag–no–square triangulations of 3–dimensional manifolds. We prove that any 3–dimensional polyhedral complex (in particular, any 3–manifold) can be triangulated in a flag–no–square way.

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1 Introduction

Gromov boundary of a word–hyperbolic group is known to be a compact finite dimensional metrizable space. It is connected unless the group essentially splits (as an amalgamated free product or as HNN-extension) over a finite subgroup. When the boundary is connected, it has no local cut point unless the group essentially splits over a two-ended subgroup [1].

Not many explicit topological spaces are known to be homeomorphic to the boundary of a word-hyperbolic group. Restricting to the case of indecomposable groups, we may ask for such spaces that are connected and have no local cut points. In dimension 1 there are precisely two possibilities: the Sierpiński carpet $M_{1,2}$ and the Menger curve $M_{1,3}$ [14]. Surprisingly little is known in dimensions above 1. The only known examples, apart from spheres S^n and Sierpiński compacta $M_{n,n+1}$, are Menger universal compacta $M_{2,5}$ and

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$M_{3,7}$ [6], and certain 2–dimensional compacta Π_p called Pontriagin surfaces [5].

Spheres (and Sierpiński compacta) occur as boundaries of fundamental groups of hyperbolic manifolds (with totally geodesic boundary). The Menger compacta $M_{2,5}$ and $M_{3,7}$ occur as boundaries of right–angled hyperbolic buildings of dimension 3 and 4 respectively. Consequently, they are the boundaries of discrete cocompact automorphism groups of such buildings. Pontriagin surfaces Π_p (where p is a prime) are characterized by $\dim_Q(\Pi_p) = \dim_{Z_q}(\Pi_p) = 1$ for any prime q distinct from p, and $\dim_{Z_p}(\Pi_p) = 2$, where $\dim_G$ is the cohomological dimension for coefficients G. They are the boundaries of certain right–angled hyperbolic Coxeter groups.

Dranishnikov's method [5] combined with a recent result of Fischer [7] yields also the Pontriagin sphere (which is different from the Pontriagin surfaces Π_p). It occurs as the boundary of all right–angled hyperbolic Coxeter groups whose nerves are closed orientable surfaces (see Remarks 3.6 and 4.4(1)).

In this paper we indicate another family of topological spaces, in dimension 3, that are the boundaries of right–angled hyperbolic Coxeter groups. These are some of the *trees of manifolds* (named so in [8]) introduced by Jakobsche in [10]. Apart from being connected, these spaces are homogeneous, and thus have no local cut points. Moreover, they are (examples of) the Cantor manifolds, which means that no subset of codimension 2 or more (i.e. of dimension ≤ 1) separates them.

Trees of manifolds are defined as inverse limits of appropriate systems of iterated connected sums of manifolds, see Section 3. They generalize the Pontriagin sphere which is obtained in this way out of 2–dimensional tori. By the already mentioned result of Fischer [7], certain trees of manifolds occur as boundaries of those right–angled hyperbolic Coxeter groups whose nerves are manifolds, PL–triangulated in flag–no–square way. An essential part of this paper is the construction of flag–no–square triangulations for any 3–dimensional polyhedral complex, in particular for any 3–dimensional PL–manifold. This construction uses certain properties of the 600–cell, and occupies Section 2.

Our approach cannot be widely extended to higher dimensions. It is known for example that 4–dimensional homology spheres do not admit flag–no–square triangulations (see Section 2.2 in [12] and Theorem 5.6 in the appendix). As a consequence, no manifold in dimension above 4 has a flag-no-square PL triangulation. The question which 4–manifolds admit flag–no–square triangulations is (according to our knowledge) an open problem. We indicate a class of such 4-manifolds in Remark 4.4(2).

On the other hand, our construction of flag–no–square triangulations in dimension 3 waits for other applications. For example, it might be useful for constructing word–hyperbolic groups whose boundaries have exotic cohomological dimensions for various coefficients, in the spirit of [5].

We close the paper with an appendix containing the summary of what is known about flag–no–square triangulations.

We are grateful to Tadeusz Januszkiewicz for posing the problem of flag–no–square subdivision of 3–dimensional simplicial complexes and for motivating us. We thank Paweł Krupski for pointing to us various topological references.

2 Flag-no-square subdivision in dimension 3

In this section we show that any simplicial complex of dimension ≤ 3 can be subdivided to satisfy flag–no–square property. We recall the definition of flag–no–square property, and briefly outline its geometric role, in the appendix to this paper.

Denote by X_{600} the boundary of the 600–cell (see e.g. [2]), a 3–dimensional euclidean simplicial polyhedron homeomorphic to the 3–dimensional sphere. It consists of 600 3–dimensional cells and 120 vertices. Its vertex links are icosahedra and edge links are pentagons. We will only exploit the combinatorial simplicial structure of X_{600}.

A starting point for our construction of flag–no–square subdivisions is the following.

Lemma 2.1. *X_{600} satisfies flag–no-square property.*

Proof. One possible argument goes by direct inspection. Namely, to see flagness, note that any collection of vertices in X_{600} that are pairwise connected with edges is contained in the star of any of those vertices. Moreover, star of any vertex is a full subcomplex in X_{600} and, since it is a simplicial cone over the icosahedron, it is flag. Thus, any such collection of vertices spans a simplex in the star of any vertex in this collection.

To see flag–no–square property, note that if the distance between two vertices x, y in the 1–skeleton of X_{600} is equal 2, there are only two possibilities for the relative position of x and y. Either there is only one point at distance 1 from both x and y or there are exactly three points with this property and they form a 2 dimensional simplex $M(x, y)$ of X_{600}. Thus, if we have a cycle of length four in the 1–skeleton of X_{600} and one pair x, y of

the opposite vertices of this cycle is not joined by an edge, then the distance between x and y is 2 and the second pair belongs to $M(x,y)$, so it is joined by an edge. This completes the first argument.

We present also another (less elementary) argument that refers to Moussong's characterization of word–hyperbolicity for right angled Coxeter groups (see Corollary 5.3 in the appendix). Consider the 120-cell P_{120}, which is a simple convex 4-dimensional polytope whose 3-dimensional faces are dodecahedra. Realize P_{120} as a right-angled convex polytope in the hyperbolic space H^4, and consider the right–angled Coxeter group W generated by reflections with respect to its 3-dimensional faces. Clearly, W is then word–hyperbolic. Its nerve is the simplicial complex dual to the boundary complex ∂P_{120}, and thus it is isomorphic to X_{600}. By the above mentioned Moussong's characterization, X_{600} is flag–no–square, and the lemma follows. □

Before constructing flag–no–square subdivisions in dimension 3, we recall the analogous construction in dimension 2, due to Dranishnikov [5]. Let Z_{10} be the subcomplex of the boundary triangulation of icosahedron which is the span of the vertices at distance at most 1 (in the 1–skeleton) from a fixed 2–dimensional simplex. Clearly, Z_{10} is topologically a 2-dimensional disc. We will call *true vertices* of Z_{10} those vertices which belong to exactly two different 2–dimensional simplices of Z_{10}. There are three such vertices and they lie on the boundary of Z_{10}.

Definition 2.2. For a 2–simplex Δ we will call the subdivision of Δ isomorphic to Z_{10}, where vertices of Δ correspond to true vertices of Z_{10}, *the special subdivision of* Δ. We will denote it by Δ^*. For any 2-dimensional simplicial complex Y, the *special subdivision* Y^* is obtained by taking the first barycentric subdivision of the 1-skeleton of Y, followed by the special subdivision of every 2-simplex of Y.

We recall from [5] the following.

Lemma 2.3. *Let Y be a 2-dimensional simplicial complex. Then its special subdivision Y^* satisfies flag–no–square property.*

Now we turn to looking closely at the combinatorial ball around a 3–simplex in X_{600}. Its properties together with the flag–no-square property of X_{600} are crucial for the later construction in this section. Fix a 3–dimensional simplex Δ_0 in X_{600}. Consider the subcomplex

$$B_1(\Delta_0) = \text{span}\{v \in X_{600} : \text{dist}(v, \Delta_0) \leq 1\}.$$

Denote by $\partial B_1(\Delta_0)$ the subcomplex in $B_1(\Delta_0)$ spanned by the vertices not in Δ_0.

Lemma 2.4. *Every simplex in $\partial B_1(\Delta_0)$ is contained in a simplex intersecting Δ_0.*

Proof. If the simplex of $\partial B_1(\Delta_0)$ is a vertex, the assertion is trivial. If the simplex is an edge xy and x', y' are neighbours in Δ_0 of x, y respectively, then flag–no–square property of X_{600} implies, w.l.o.g., that xy' is an edge, so xy belongs to a triangle xyy' intersecting Δ_0.

Now if xyz is a triangle in $\partial B_1(\Delta_0)$ then let x', y' denote vertices in Δ_0 forming triangles zxx', zyy' guaranteed by the previous step. If $x' = y'$ then the assertion follows by flagness of X_{600}. If not, consider the cycle $xyy'x'x$ and note that the flag–no–square condition yields, w.l.o.g., that xy' is an edge, so $xyzy'$ is a simplex.

Finally, we need to show that $\partial B_1(\Delta_0)$ contains no 3–simplex. If not, let x, y, z, t be vertices spanning a 3–simplex of $\partial B_1(\Delta_0)$. Let u, w be the vertices in Δ_0 forming 3–simplices $xyzu$ and $yztw$ guaranteed by the previous step. If $u = w$ then flagness of X_{600} yields that $xyztuw$ is a 4–simplex of X_{600}, a contradiction. If $u \neq w$, consider the cycle $uwtxu$ and note that the flag–no–square condition yields, w.l.o.g., that xw is an edge, so $xyzuw$ is a 4–simplex of X_{600}, which gives again a contradiction. □

The above lemma allows to describe both complexes $B_1(\Delta_0)$ and $\partial B_1(\Delta_0)$ more precisely.

Lemma 2.5.

(1) $B_1(\Delta_0)$ is topologically a 3-ball, and $\partial B_1(\Delta_0)$ is its boundary sphere.

(2) $\partial B_1(\Delta_0)$ is simplicially isomorphic to the special subdivision $(\partial\sigma^3)^*$ of the boundary of a 3-simplex.

(3) Under the canonical identification of $\partial B_1(\Delta_0)$ with $(\partial\sigma^3)^*$, if vertices $w_1, w_2 \in \partial B_1(\Delta_0)$ are contained in the same proper face τ of σ^3, then there is a vertex $v_0 \in \Delta_0$ at distance 1 from both w_1 and w_2.

Proof. Denote the vertices of Δ_0 by $v_i,\ i = 1, 2, 3, 4$. First consider four 3–dimensional simplices $\Delta_i,\ i = 1, 2, 3, 4$ of X_{600}, which have 2–dimensional intersection with Δ_0 ($v_i \notin \Delta_0 \cap \Delta_i$). Denote by $a_i,\ i = 1, 2, 3, 4$ the vertices in corresponding Δ_i not contained in Δ_0, thus being vertices in $\partial B_1(\Delta_0)$.

Now, consider twelve 3–dimensional simplices $\Delta_{ij},\ i, j = 1, 2, 3, 4,\ i \neq j$, such that $\Delta_{ij} \cap \Delta_0$ is 1–dimensional, $\Delta_{ij} \cap \Delta_0 = \Delta_i \cap \Delta_j,\ \Delta_{ij} \cap \Delta_i$ is 2–dimensional. In each Δ_{ij} there is one edge disjoint with Δ_0, whose one vertex is a_i and second is a common vertex with Δ_{ji}. Denote this common vertex by a_{ij} $(i < j)$. So together, the union $\Delta_0 \cup \bigcup \Delta_i \cup \bigcup \Delta_{ij}$ is homeomorphic to

the 3-ball, and its intersection with $\partial B_1(\Delta_0)$ is simplicially isomorphic to the first barycentric subdivision of the 1–skeleton of the 3–dimensional simplex.

Finally, consider the rest of the 3–dimensional simplices of X_{600} having non–empty intersection with Δ_0, grouped into the following four complexes. Let

$$C_i = \{\bigcup \Delta : \Delta \text{ is a 3–dimensional simplex of } B_1(\Delta_0),\ \Delta \cap \Delta_0 = v_i\},$$

i=1,2,3,4. Recall that the star of the vertex v_i is the cone over icosahedron. Observe that C_i consists of exactly half of 3–dimensional simplices of this star. To see this, we check which simplices of this star do not belong to C_i. First, C_i does not contain Δ_0. It also does not contain any of $\Delta_j,\ j \neq i$, since these have 2–dimensional intersection with Δ_0. Finally, C_i does not contain the simplices $\Delta_{kl},\ k, l \neq i$, since they have 1–dimensional intersection with Δ_0. It particular, the triangles corresponding to these 3–simplices in the link of v_i are the triangles which have a nonempty intersection with the triangle corresponding to Δ_0. Thus the intersection of C_i with the link of v_i (which is exactly the intersection with $\partial B_1(\Delta_0)$) is simplicially isomorphic with Z_{10}. It is glued to the previously constructed part of $\partial B_1(\Delta_0)$ in such a way that true vertices of the new part are glued to the vertices $a_j,\ j \neq i$. This completes the proof of part (2).

Part (1) follows by observing that each C_i is topologically a 3–ball, glued to the previously constructed part of $B_1(\Delta_0)$ (which is itself a 3–ball) along a 2–disk contained in the boundary ∂C_i. The direct inspection yields also that the subcomplex $\partial B_1(\Delta_0)$ is the boundary sphere of the so obtained 3–ball.

Part (3) easily follows from the above description of $B_1(\Delta_0)$. □

In order to extend special subdivisions to dimension 3, we will use the complement of the ball $B_1(\Delta_0)$ in X_{600}. More precisely, denote by X_{543} the 3–dimensional subcomplex of X_{600} which is the closure of $X \backslash B_1(\Delta_0)$. In other words, X_{543} is the union of all 3–simplices of X_{600} not contained in $B_1(\Delta_0)$. (The number 543 in the subscript is the number of 3-simplices in this subcomplex.)

Remark 2.6. It follows from Lemma 2.4 that X_{543} is a 3-ball and its boundary sphere is simplicially isomorphic to $(\partial\sigma^3)^*$.

Lemma 2.7.

(1) X_{543} is a full subcomplex of X_{600}.

(2) X_{543} satisfies flag–no–square property.

Proof. To prove (1), consider a collection of vertices in X_{543} spanning a simplex τ of X_{600}. If at least one of those vertices is interior in X_{543} then

clearly τ is a simplex of X_{543}. Otherwise, suppose that all simplices in the collection belong to the boundary sphere $\partial B_1(\Delta_0)$. Since, by definition, the subcomplex $\partial B_1(\Delta_0)$ is full in X_{600}, the assertion (1) follows.

Since flag–no–square property is inherited by full subcomplexes (Lemma 5.1 in Appendix), assertion (2) follows from Lemma 2.1. □

Now we are ready to define special subdivision for 3-dimensional complexes.

Definition 2.8. Given a simplicial complex W with $\dim W \leq 3$, its *special subdivision* W^* is the simplicial complex obtained by taking the special subdivision of the 2–skeleton $W^{(2)}$ followed by subdividing each 3–simplex σ^3 of W so that it becomes isomorphic to X_{543} and its subdivided boundary $(\partial\sigma^3)^*$ canonically identifies with $\partial B_1(\Delta_0)$.

To prove that special subdivision yields simplicial complexes that satisfy flag–no–square property, we need some preparatory results.

Lemma 2.9. *Let Δ be a simplex of dimension 1, 2 or 3, and let xy be an edge of the special subdivision Δ^*. If the vertices x, y are not both contained in a common proper face of Δ, then at least one of them lies in the interior of Δ.*

Proof. For dimension 1 and 2 the proof goes by inspection. For dimension 3, suppose that both x and y are contained in the boundary of Δ. Identify Δ^* with $X_{543} \subset X_{600}$, and recall that its boundary $\partial B_1(\Delta_0)$ is by definition full in X_{600}. Thus the edge xy is contained in the boundary too. Let τ be a proper face of Δ containing the edge xy. Then both x and y are contained in τ, a contradiction. □

Lemma 2.10. *Let W be a simplicial complex with $\dim W \leq 3$ and let U be a subcomplex of W. Then U^* is a full subcomplex of W^*.*

Proof. Let vertices $x, y \in U^*$ form an edge xy of W^*. Denote by $\Delta \in W$ the simplex of the lowest possible dimension such that $xy \in \Delta^*$. By Lemma 2.9, at least one of the vertices, say y, belongs to the interior of Δ. This implies that $\Delta \in U$ and $xy \in \Delta^* \subset U^*$.

For a triangle or a 3–simplex σ of W^* with vertices in U^* the argument is the same. If Δ is a minimal simplex of W such that $\sigma \subset \Delta$, then at least one of the vertices y of σ belongs to the interior of Δ, so $\Delta \in U$ and $\sigma \in \Delta^* \subset U^*$. □

Lemma 2.11. *Let Δ be a 3–dimensional simplex. Let w, w_1, w_2 be vertices of Δ^* and suppose w is interior in Δ and $w_1, w_2 \in \partial\Delta^*$. Suppose also that w_1, w_2 lie on the same 2–dimensional face of Δ, and that there are edges in Δ^* between w and w_1 and between w and w_2. Then there is an edge between w_1 and w_2 in Δ^*.*

Proof. View Δ^* as X_{543}, a subcomplex of X_{600}. By Lemma 2.5(3), there is a vertex v_0 in Δ_0 whose distance from both w_1 and w_2 is equal to one. If w_1 and w_2 were not connected by an edge, then the opposite vertices of the cycle $w_1v_0w_2ww_1$ wouldn't be connected with edges, contradicting flag–no–square property of X_{600}. □

Lemma 2.12. *Let Δ be a 3–simplex and τ a simplex of dimension ≤ 3 such that $\Delta \cap \tau$ is a nonempty proper face in both Δ and τ. Then the special subdivision $Y = (\Delta \cup \tau)^*$ satisfies flag–no–square property.*

Proof. To prove flagness, consider any collection of vertices in Y pairwise connected with edges. We claim all these vertices belong to a single subdivided simplex Δ^* or τ^*. Indeed, if one of the vertices, say v, is not in the intersection $\Delta \cap \tau$ then all vertices in the collection belong to the same subdivided simplex as v. Otherwise, all vertices in the collection are contained in both simplices Δ and τ. Now, since each of the subdivided simplices Δ^* and τ^* is full in Y and flag, the vertices from the collection span a simplex of Y.

To prove flag–no–square property, suppose we have a cycle of length four in Y. If the cycle is contained in Δ^* or in τ^*, then we use the flag–no–square property of X_{543} (Lemma 2.7(2)), or Lemma 2.3. Otherwise, suppose that a vertex x of the cycle does not belong to Δ^*, and that a vertex y does not belong to τ^*. This means, that x and y are opposite in the cycle and that the other two vertices, u and w, of the cycle belong to $\Delta \cap \tau$. By Lemma 2.11, u and w are connected with an edge, which finishes the proof. □

We are now ready to prove the main result of this section, the proposition below.

Proposition 2.13. *Let W be a 3–dimensional simplicial complex. Then its special subdivision W^* is flag–no–square.*

Proof. To prove flagness, consider a collection of vertices of W^* pairwise connected with edges. If all these vertices are in $(W^{(2)})^*$, the special subdivision of the 2-skeleton of W, then they are pairwise connected with edges in $(W^{(2)})^*$ by the fact that $(W^{(2)})^*$ is full in $W*$ (Lemma 2.10). They span

a simplex by the fact that $(W^{(2)})^*$ is flag (Lemma 2.3). Otherwise, there is a vertex in the collection contained in the interior of a 3-simplex Δ of W. Consequently, all vertices in the collection are the vertices of Δ^*. Now, since Δ^* is full in W^* (Lemma 2.10), the vertices in the collection are pairwise connected with edges in Δ^*. Since the latter is flag (Lemma 2.7(2)), the vertices span a simplex of Δ^*, which is also a simplex of W^*.

To prove flag–no–square property, consider a cycle of length 4 in W^*. If all vertices of the cycle are in $(W^{(2)})^*$ then, by Lemma 2.10, the whole cycle is contained in $(W^{(2)})^*$. Since the latter is flag–no–square (Lemma 2.3), there is an edge between some opposite vertices is the cycle.

If one vertex, say v_1, in the cycle $v_1v_2v_3v_4v_1$ in W^* is interior in a 3-simplex Δ of W, then v_2 and v_4 are together with v_1 in Δ^*. If v_3 is also in Δ^*, the whole cycle is in Δ^* (by fullness of Δ^* in W^*, see Lemma 2.10). Since Δ^* is flag–no–square, the cycle is as required. If v_3 is not contained in Δ, it belongs to some simplex τ which is not a face of but shares a face with Δ. Moreover, the whole cycle is contained in the union $\Delta \cup \tau$ (Lemma 2.10 again). Since the special subdivision of the latter is flag–no–square (Lemma 2.12), this finishes the proof. □

Corollary 2.14. *Every 3–dimensional polyhedron can be triangulated in a flag-no-square fashion.*

Proof. Consider any triangulation of the polyhedron and take its special subdivision. □

3 Trees of manifolds

In this section we recall from [10] Jakobsche's definition of a family of spaces that we call (after Fischer and Guilbault [8]) *trees of manifolds*. We include an extension to the case of nonorientable manifolds due to P. Stallings [19]. We recall and/or derive some useful topological properties of these spaces. We also recall a theorem of Fischer [7] who proved that some trees of manifolds are CAT(0) boundaries of the Davis–Vinberg complexes of certain right–angled Coxeter groups. We focus only on trees of manifolds, which appear in Fischer's theorem.

Theorem 3.1 (Jakobsche [10], [7], Stallings [19])**.** *Let $L_0 \overset{\alpha_1}{\leftarrow} L_1 \overset{\alpha_2}{\leftarrow} L_2 \overset{\alpha_3}{\leftarrow} \dots$ be an inverse sequence of connected closed n–manifolds ($n \geq 2$) and $\mathcal{D}_k$ finite collections of disjoint collared disks in L_k such that*

(a) each L_k is a connected sum of finitely many copies of L_0;

(b) each α_{k+1} is a homeomorphism over the set $L_k \setminus \bigcup\{\text{int } D | D \in \mathcal{D}_k\}$;

(c) each $\alpha_{k+1}^{-1}(D)$ $(D \in \mathcal{D}_k)$ is homeomorphic to a copy of L_0 with the interior of a collared disk removed;

(d) $\{\alpha_{j+1} \circ \alpha_{j+2} \circ \ldots \circ \alpha_i(D) | D \in \mathcal{D}_i, i \geq j\}$ is null and dense in L_j for all j;

(e) $\alpha_{j+1} \circ \alpha_{j+2} \circ \ldots \circ \alpha_i(D) \cap \text{bdy} D' = \emptyset$ for all $D \in \mathcal{D}_i, D' \in \mathcal{D}_j, i > j$.
Then the inverse limit

$$\lim_{\leftarrow}(L_0 \overset{\alpha_1}{\leftarrow} L_1 \overset{\alpha_2}{\leftarrow} L_2 \overset{\alpha_3}{\leftarrow} \ldots)$$

depends on L_0 only. This space is denoted by $X(L_0, \{L_0\})$.

The spaces $X(L_0, \{L_0\})$ are clearly connected and locally connected compact metric spaces. Jakobsche and Stallings show in [11], [10] and [19] the following less immediate properties of these spaces. Recall that, given a positive integer m, a topological space X is m–homogeneous if for any two m–element subsets of X there is a homeomorphism of this space which maps one set to the other.

Theorem 3.2 (W. Jakobsche, P. Stallings)**.**

(1) Topological dimension $\dim X(L_0, \{L_0\})$ is equal to $\dim L_0$.

(2) For every positive integer m the space $X(L_0, \{L_0\})$ is m–homogeneous.

(3) If L_0 is a homology n–sphere then $X(L_0, \{L_0\})$ is a cohomology n–manifold.

The above properties of trees of manifolds have the following further consequences, especially interesting in the context of boundaries of word–hyperbolic groups. Recall that a topological space X is a *Cantor manifold* if no subset of X of dimension $\leq \dim X - 2$ separates X.

Corollary 3.3. *The spaces $X(L_0, \{L_0\})$*

(1) have no local cut points, and

(2) are Cantor manifolds.

Proof. Let $X = X(L_0, \{L_0\})$. By homogeneity, either X has no local cut point or every point of X is a local cut point. Suppose the latter holds. Since in any continuum the set of local cut points which are not of order 2 is countable ([20], (9.2), p. 61), it follows that each point of X is of order 2, and hence it is homeomorphic to S^1 ([16], Theorem 6, p. 294). This contradiction proves (1).

Since any homogeneous continuum is a Cantor manifold [15], (2) follows from Theorem 3.2(2). □

It turns out that sometimes the spaces $X(L_0, \{L_0\})$ are homeomorphic for different manifolds L_0. On the other hand, Jakobsche [10] shows how to distinguish certain trees of manifolds, up to homeomorphism, in dimension 3. We recall briefly some details concerning these two issues. We denote by $M\#N$ the connected sum of manifolds M and N.

Lemma 3.4. *If $M' = M\#N$ and $M = L\#N$ then the spaces $X(M', \{M'\})$ and $X(M, \{M\})$ are homeomorphic.*

Proof. Both spaces are easily seen to be homeomorphic to the Jakobsche's space $X(L, \{L, N\})$ (see [10] for the definition). □

In dimension 2, in view of the classification of surfaces, Lemma 3.4 implies the following.

Corollary 3.5. *Let F_1, F_2 be closed surfaces different from the 2–sphere. If either both F_1, F_2 are orientable or both are non–orientable, then the spaces $X(F_1, \{F_1\})$ and $X(F_2, \{F_2\})$ are homeomorphic.*

Remark 3.6. Note that if F is an orientable closed surface then the space $X(F, \{F\})$ is the well known Pontriagin sphere. If F is a non–orientable closed surface then $X(F, \{F\})$ is the Pontriagin surface Π_2 mentioned in the introduction (see [21]). Pontriagin sphere is not homeomorphic to any of the Pontriagin surfaces Π_p, as its cohomological dimension for coefficients Q equals 2.

In the case of 3–manifolds, Lemma 3.4 and the argument as in the proof of Theorem 11.1 in [10] imply the following.

Proposition 3.7.

(1) If L_0, L_0' are closed 3–manifolds different from the sphere, and the summands appearing in their prime decompositions coincide (the numbers of their occurences do not have to match) then the spaces $X(L_0, \{L_0\})$ and $X(L_0', \{L_0'\})$ are homeomorphic.

(2) Let L_0, L_0' be closed orientable 3–manifolds. Suppose that, in their prime decompositions, certain summand of L_0 has different fundamental group from all summands of L_0'. Then the spaces $X(L_0, \{L_0\})$ and $X(L_0', \{L_0'\})$ are not homeomorphic.

Now let us state the announced theorem of Fischer. For the definition of the Davis-Vinberg complex of a Coxeter group see [3]. By bdy(Γ) we denote the CAT(0) boundary of the Davis–Vinberg complex of the group Γ.

Theorem 3.8 (Fischer [7]). *If the nerve N of a right–angled Coxeter group Γ is a connected closed (orientable or not) PL manifold, then* $\mathrm{bdy}(\Gamma)$ *is homeomorphic to Jakobsche's $X(|N|, \{|N|\})$ space.*

4 The main theorem

In this section we formulate and prove the main result of the paper, Theorem 4.1. We make comment on the algebraic consequences for groups appearing in Theorem 4.1, implied by topological properties of 3–dimensional trees of manifolds. Finally, we show the consequences of the arguments similar to ours for dimensions different than 3.

Recall that all 3-manifolds are triangulable, all their triangulations are PL, and any two triangulations of a fixed 3-manifold are PL equivalent [17].

Theorem 4.1. *Let N be a connected closed 3–dimensional manifold. Then there exists a right–angled Coxeter group Γ which is word–hyperbolic and its Gromov boundary is homeomorphic to Jakobsche's $X(N, \{N\})$ space.*

Proof. Take any *PL* triangulation of N. Let N^* be its special subdivision, as defined in Section 2. Since N^* is flag–no–square (Proposition 2.13), the right angled Coxeter group Γ whose nerve is N^* is word–hyperbolic (Lemma 5.3 in the appendix). The Gromov boundary of Γ is homeomorphic to the CAT(0) boundary of the Davis–Vinberg complex of Γ. Thus, due to Fischer's Theorem 3.8, the Gromov boundary of Γ is homeomorphic to Jakobsche's $X(N, \{N\})$ space. □

Remark 4.2. Since the space $X(N, \{N\})$ is connected and has no local cut point (Lemma 3.3(1)), it follows from a result of Bowditch [1] that Γ as in Theorem 4.1 is JSJ-indecomposable (i.e. it does not split as amalgamated free product or HNN extension over a finite or 2–ended subgroup). Moreover, since $X(N, \{N\})$ is a Cantor manifold of topological dimension 3 (Lemma 3.3(2) and Theorem 3.2(1)), it is not separated by a Cantor set or a circle. Consequently, the corresponding group Γ does not split over an undistorted virtually free or virtually surface subgroup.

The argument as in the proof of Theorem 4.1 clearly gives the following.

Corollary 4.3. *If N is a connected closed manifold (of arbitrary dimension greater than 1) which admits a flag–no–square PL triangulation then the space $X(N, \{N\})$ is homeomorphic to Gromov boundary of a word-hyperbolic right-angled Coxeter group.*

Remark 4.4.

(1) The above corollary, together with Dranishnikov's Lemma 2.3, show that there are word–hyperbolic groups whose boundaries are homeomorphic to the Pontriagin sphere.

(2) In dimension 4 the only known to us examples of manifolds N as in Corollary 4.3 are the following. Consider the regular simplicial tesselation of the hyperbolic space H^4 with all vertex links isomorphic to X_{600} – the boundary complex of the 600–cell. It is not hard to show (using Lemma 2.1) that this tesselation is flag–no–square. The automorphism group G of this tesselation is a Coxeter group, hence it is residually finite. In particular, G contains torsion–free subgroups G' for which the quotients have arbitrarily large injectivity radius. If the injectivity radius is large enough, the quotient (viewed as a simplicial manifold) is flag–no–square. It is PL regardless the injectivity radius.

(3) It follows from a result of Januszkiewicz and Świątkowski [12] (see also Corollary 5.7(2) in the appendix) that no closed manifold of dimension ≥ 5 admits a flag–no–square triangulation. Thus Corollary 4.3 gives no trees of manifolds in dimensions above 4 as Gromov boundaries of word-hyperbolic groups.

5 Appendix: Flag-no-square triangulations

The aim of this appendix is to survey the subject of flag-no-square triangulations, and to formulate some open questions. All we know about this we have learnt from or discovered together with Tadeusz Januszkiewicz.

For completeness, we start with definitions.

A simplicial complex X is *flag* if any finite subset of its vertices pairwise connected with edges spans a simplex X.

A *cycle* in X is a subcomplex homeomorphic to the circle S^1. The *length* of a cycle is the number of edges in this cycle. A *diagonal* in a cycle is an edge connecting any two non-consecutive vertices in this cycle.

Simplicial complex X is said to satisfy *flag–no–square* condition if X is flag and any cycle of length 4 in X has a diagonal (equivalently, there is no full cycle of length 4 in X). We shortly say that X *is* flag–no–square. The same condition is also known as "no empty square" condition.

The following two properties of flag–no–square condition are immediate but useful consequences of the definition. The second property follows from

the first by the fact that links in a flag simplicial complex are its full subcomplexes.

Lemma 5.1. *Let X be a flag–no–square simplicial complex. Then:*

(1) any full subcomplex of X is flag-no-square;

(2) links of X are flag–no–square.

Flag–no–square condition was introduced and studied in the context of cubical structures on 3-manifolds by L. Siebenmann. Its importance for the geometry of cubical complexes comes from the following observation by Gromov ([9], p. 123).

Proposition 5.2. *Let X be a cubical complex with flag–no–square links. Then X admits a negatively curved piecewise hyperbolic metric. In particular, the fundamental group of X is word–hyperbolic.*

This proposition implies in particular the following (see [18] or [4]).

Corollary 5.3. *A right–angled Coxeter group is word–hyperbolic iff its nerve is a flag–no–square simplicial complex.*

In [13] flag–no–square condition has been reintroduced under the name 5–*largeness*, and put in the wider context of combinatorial "metric" conditions for simplicial complexes. In this approach, we say that a simplicial complex X is k–large (for an integer $k \geq 4$) if X is flag and any cycle γ in X, with length $|\gamma|$ satisfying $3 < |\gamma| < k$, has a diagonal (equivalently, any full cycle in X has length $\geq k$).

It turns out that 6–largeness of links (called shortly *local 6–largeness*) is a condition that resembles nonpositive curvature, sharing many consequences with the latter. For example, any compact locally 6–large simplicial complex is aspherical and its fundamental group is semi–hyperbolic. Furthermore, the fundamental group of any compact locally 7–large simplicial complex is word–hyperbolic.

Local 5–largeness is a weaker condition and does not lead to phenomena related to nonpositive curvature. For example, boundaries of the dodecahedron and the 600–cell are 5–large (and thus also locally 5–large) triangulations of the spheres S^2 and S^3. However, this condition may be still viewed as a kind of upper curvature bound. There are reasons to expect topological consequences of local 5–largeness similar to those of nonpositive curvature, in higher dimensions.

In the remaining part of this appendix we will use the term "5–large" instead of "flag–no–square".

In low dimensions there are no topological restrictions for 5–largeness. Namely, the following fact was observed by Dranishnikov [5].

Proposition 5.4. *Any 2–dimensional polyhedron admits a 5–large triangulation.*

In this paper we have strengthened this result by showing

Proposition 5.5. *Any polyhedron of dimension* ≤ 3 *admits a 5–large triangulation.*

Starting from dimension 4, there are topological obstructions for 5–largeness. They concern, among others, the so called generalized homology spheres (GHS).

A simplicial complex N is a *generalized homology sphere* of dimension k if it has the same homology as the sphere S^k and if the link of each simplex in N is a generalized homology sphere of appropriate dimension. Along the lines of Section 2.2 in [12] the following result is proved.

Theorem 5.6. *A generalized homology sphere of dimension* $k \geq 4$ *is never 5–large.*

Recall that links of PL–triangulations of manifolds are spheres. Links of other triangulations of manifolds may not be spheres, but they are generalized homology spheres. In particular, all triangulations of spheres and homology spheres are GHS. Moreover, links in any triangulation of a manifold are GHS. In view of these facts Theorem 5.6 has the following consequences.

Corollary 5.7.

(1) No triangulation of a 4–dimensional homology sphere is 5–large. In particular, no triangulation of the sphere S^4 *is 5–large.*

(2) No triangulation of a manifold of dimension $n \geq 5$ *is 5–large.*

In contrast with the above result, constructions in [12] and [13] give many examples of compact 5–large pseudomanifolds, in arbitrary dimension. In fact, the examples constructed there are 6–large, and thus aspherical. When applied to cubical structure of the Davis' complex of a right angled Coxeter group, those examples yield word-hyperbolic Coxeter groups with arbitrarily large virtual cohomological dimension [12].

Coming back to manifolds, the only known to us examples of closed simplicial flag–no–square 4-manifolds are those described in Remark 4.4(2) in this paper.

We finish by posing some open problems which we find intriguing in the perspective of viewing local 5–largeness as certain upper curvature bound.

Questions 5.8.

(1) Is every 5–large (or locally 5–large) 4–dimensional simplicial manifold aspherical?

(2) Is every 5–large (or locally 5–large) n–dimensional simplicial pseudomanifold, for $n \geq 4$, aspherical?

(3) Find any restrictions for polyhedra in dimensions ≥ 4 to admit 5–large or locally 5–large triangulation.

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