INFINITE GROUPS WITH FIXED POINT PROPERTIES

G. ARZHANTSEVA, M.R. BRIDSON, T. JANUSZKIEWICZ, I.J. LEARY, A. MINASYAN,
AND J. ŚWIĄTKOWSKI

Dedicated to Michael W. Davis on the occasion of his 60th birthday.

ABSTRACT. We construct finitely generated groups with strong fixed point properties. Let $\mathcal{X}_{ac}$ be the class of Hausdorff spaces of finite covering dimension which are mod-p acyclic for at least one prime p. We produce the first examples of infinite finitely generated groups Q with the property that for any action of Q on any $X \in \mathcal{X}_{ac}$, there is a global fixed point. Moreover, Q may be chosen to be simple and to have Kazhdan's property (T). We construct a finitely presented infinite group P that admits no non-trivial action by diffeomorphisms on any smooth manifold in $\mathcal{X}_{ac}$. In building Q, we exhibit new families of hyperbolic groups: for each $n \geq 1$ and each prime p, we construct a non-elementary hyperbolic group $G_{n,p}$ which has a generating set of size $n + 2$, any proper subset of which generates a finite p-group.

1. INTRODUCTION

We present three templates for proving fixed point theorems; two are based on relative small cancellation theory and one is based on the Higman Embedding Theorem. Each template demands as input a sequence of groups with increasingly strong fixed point properties. By constructing such sequences we prove the following fixed point theorems.

For p a prime, one says that a space is *mod-p acyclic* if it has the same mod-p Čech cohomology as a point. Let $\mathcal{X}_{ac}$ be the class of all Hausdorff spaces X of finite covering dimension such that there is a prime p for which X is mod-p acyclic. Let $\mathcal{M}_{ac}$ denote the subclass of smooth manifolds in $\mathcal{X}_{ac}$.

Note that the class $\mathcal{X}_{ac}$ contains all finite dimensional contractible spaces and all finite dimensional contractible CW-complexes.

Theorem 1.1. *There is an infinite finitely generated group Q that cannot act without a global fixed point on any $X \in \mathcal{X}_{ac}$. If $X \in \mathcal{X}_{ac}$ is mod-p acyclic, then so is the fixed point set for any action of Q on X. For any countable group C, the group Q can be chosen to have either the additional properties (i), (ii) and (iii) or (i), (ii) and (iii)$'$ described below:*

   (i) *Q is simple;*
   (ii) *Q has Kazhdan's property (T);*
   (iii) *Q contains an isomorphic copy of C;*
   (iii)$'$ *Q is periodic.*

Since a countable group can contain only countably many finitely generated subgroups, it follows from property (iii) that there are continuously many (i.e., $2^{\aleph_0}$) non-isomorphic groups Q with the fixed point property described in Theorem 1.1.

Note that Kazhdan's property (T) of a countable group is equivalent to the fact that every isometric action of the group on a Hilbert space has a global fixed point.

No non-trivial finite group has a fixed point property as strong as the one in Theorem 1.1. Any finite group not of prime power order acts without a global fixed point on some finite dimensional contractible simplicial complex. Smith theory tells us that the fixed point set for any action of a finite p-group on a finite dimensional mod-p acyclic space is itself mod-p acyclic, but it is easy to construct an action of a non-trivial finite p-group on a 2-dimensional mod-q acyclic space without a global fixed point if q is any prime other than p. Since the fixed point property of Theorem 1.1 passes to quotients, it follows that none of the groups Q can admit a non-trivial finite quotient. This further restricts the ways in which Q can act on acyclic spaces. For example, if $X \in \mathcal{X}_{ac}$ is a locally finite simplicial complex and Q is acting simplicially, then the action of Q on the successive star neighbourhoods $st_{n+1} := st(st_n(x))$ of a fixed point $x \in X$ must be trivial, because st_n is Q-invariant and there is no non-trivial map from Q to the finite group $\mathrm{Aut}(st_n)$. Since $X = \bigcup_n st_n$, we deduce:

Corollary 1.2. *The groups Q from Theorem 1.1 admit no non-trivial simplicial action on any locally-finite simplicial complex $X \in \mathcal{X}_{ac}$.*

The ideas behind the above corollary can be further developed in different directions. The following was suggested to the authors by N. Monod:

Corollary 1.3. *A group Q from the claim of Theorem 1.1 admits no non-trivial isometric action on any proper metric space $X \in \mathcal{X}_{ac}$.*

Indeed, suppose that $x_0 \in X$ is fixed by the action of Q. Then we have a homomorphism $\phi : Q \to \mathrm{Isom}(X)$ such that $\phi(Q) \leq St(x_0)$, where $St(x_0)$ denotes the stabilizer of x_0 in the full group of isometries $\mathrm{Isom}(X)$ of X. Since X is proper, by a general version of Arzelà-Ascoli theorem (see [4, I.3.10]) the group $St(x_0)$ is compact (endowed with the topology of uniform convergence on compact subsets). Suppose that the action of Q on X is non-trivial. Then there is $g \in Q$ such that $h = \phi(g) \neq 1$. By Peter-Weyl

theorem (see [21, Sec. 4]) there exists a finite-dimensional irreducible representation ρ of $St(x_0)$ such that $\rho(h) \neq \text{Id}$. Hence the image $\rho \circ \phi(Q)$ is a non-trivial finitely generated linear group, which must be residually finite by a well-known result of Malcev (see [14, III.7.11]). The latter implies that Q has a non-trivial finite quotient, contradicting the assumptions.

Using similar arguments we can also rule out non-trivial real analytic actions of the groups Q (from Theorem 1.1) on any acyclic manifold M: assuming that the action fixes a point $x_0 \in M$, the image of every element of Q in the group $\text{Homeo}(M)$ can be regarded as an n-tuple of formal power series on n variables with real coefficients and trivial constant terms, where $n = \dim(M)$. Thus, such an action gives rise to a homomorphism from Q into the group H_n of such n-tuples of formal power series with respect to superposition. One can argue that H_n is embedded into an inverse limit of linear groups, therefore every finitely generated subgroup of H_n is residually finite. As before, the absence of non-trivial finite quotients for Q yields the triviality of the above homomorphism.

However, a stronger result concerning triviality of actions on manifolds can be obtained more directly:

Proposition 1.4. *A simple group G that contains, for each $n > 0$ and each prime p, a copy of $(\mathbb{Z}_p)^n$ admits no non-trivial action by diffeomorphisms on any $X \in \mathcal{M}_{ac}$. The group Q in Theorem 1.1 may be chosen to have this property.*

Finite p-groups have a global fixed point whenever they act on compact Hausdorff spaces that are mod-p acyclic, but the groups Q do not have this property. Indeed, if Q is infinite and has property (T) then it will be non-amenable, hence the natural action of Q on the space of finitely-additive probability measures on Q will not have a global fixed point, and this space is compact, contractible, and Hausdorff.

We know of no finitely presented group enjoying the fixed point property described in Theorem 1.1. However, using techniques quite different from those used to construct the groups Q, we shall exhibit finitely presented groups that cannot act on a range of spaces. In particular we construct groups of the following type.

Theorem 1.5. *There exist finitely presented infinite groups P that have no non-trivial action by diffeomorphisms on any smooth manifold $X \in \mathcal{M}_{ac}$.*

Certain of the Higman-Thompson groups [8], such as R. Thompson's vagabond group V, can also serve in the role of P (cf. Remark 4.6).

Theorem 1.1 answers a question of P. H. Kropholler, who asked whether there exists a countably infinite group G for which every finite-dimensional contractible G-CW-complex has a global fixed point. This question is motivated by Kropholler's study of the closure operator н for classes of groups, and by the class нꟊ obtained by applying this operator to the class ꟊ of all finite groups [12]. Briefly, if 𝕮 is a class of groups, then the class н𝕮 is the smallest class of groups that contains 𝕮 and has the property that if the group G admits a finite-dimensional contractible G-CW-complex X with all

stabilizers already in $\mathbf{H}\mathfrak{C}$, then G is itself in $\mathbf{H}\mathfrak{C}$. Kropholler showed that any torsion-free group in $\mathbf{H}\mathfrak{F}$ of type FP_∞ has finite cohomological dimension. Since Thompson's group F is torsion-free and of type FP_∞ but has infinite cohomological dimension [6], it follows that F is not in $\mathbf{H}\mathfrak{F}$. Until now, the only way known to show that a group is not in $\mathbf{H}\mathfrak{F}$ has been to show that it contains Thompson's group as a subgroup. If Q is any of the groups constructed in Theorem 1.1, then Q has the property that for any class $\mathfrak{C}$ of groups, Q is in the class $\mathbf{H}\mathfrak{C}$ if and only if Q is already in the class $\mathfrak{C}$. In particular, Q is not in the class $\mathbf{H}\mathfrak{F}$. Note that many of the groups constructed in Theorem 1.1 cannot contain Thompson's group F as a subgroup, for example the periodic groups.

Our strategy for proving Theorems 1.1 and 1.5 is very general. First, we express our class of spaces as a countable union $\mathcal{X} = \cup_{n\in\mathbb{N}}\mathcal{X}_n$. For instance, if all spaces in $\mathcal{X}$ are finite-dimensional, then $\mathcal{X}_n$ may be taken to consist of all n-dimensional spaces in $\mathcal{X}$. Secondly, we construct finitely generated groups G_n that have the required properties for actions on any $X \in \mathcal{X}_n$. Finally, we apply the templates described below to produce the required groups.

Template **FP**: *ruling out fixed-point-free actions.* If there is a sequence of finitely generated *non-elementary relatively hyperbolic* groups G_n such that G_n cannot act without a fixed point on any $X \in \mathcal{X}_n$, then there is an infinite finitely generated group that cannot act without a fixed-point on any $X \in \mathcal{X}$.

Template **NA**$_{fg}$: *ruling out non-trivial actions.* If there is a sequence of non-trivial finitely generated groups G_n such that G_n cannot act non-trivially on any $X \in \mathcal{X}_n$, then there is an infinite finitely generated group that cannot act non-trivially on any $X \in \mathcal{X}$.

Template **NA**$_{fp}$: *finitely presented groups that cannot act.* Let $(G_n; \xi_{n,j})$ $(n \in \mathbb{N}, j = 1, \ldots, J)$ be a recursive system of non-trivial groups and monomorphisms $\xi_{n,j} : G_n \to G_{n+1}$. Suppose that each G_{n+1} is generated by $\bigcup_j \xi_{n,j}(G_n)$ and that for every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that G_n cannot act non-trivially on any $X \in \mathcal{X}_m$. Then there exists an infinite finitely presented group that cannot act non-trivially on any $X \in \mathcal{X}$.

Only the first and third templates are used in the construction of the groups P and Q. We include the second template with a view to further applications.

The engine that drives the first two templates is the existence of common quotients established in Theorem 1.6 below. The proof of this theorem, given in Section 2, is based on the following result of Arzhantseva, Minasyan and Osin [1], obtained using small cancellation theory over relatively hyperbolic groups: any two finitely generated non-elementary relatively hyperbolic groups G_1, G_2 have a common non-elementary relatively hyperbolic quotient H.

Theorem 1.6. *Let $\{G_n\}_{n\in\mathbb{N}}$ be a countable collection of finitely generated non-elementary relatively hyperbolic groups. Then there exists an infinite finitely generated group Q that is a quotient of G_n for every $n \in \mathbb{N}$.*

Moreover, if C is an arbitrary countable group, then such a group Q can be made to satisfy the following conditions

    (i) *Q is a simple group;*
   (ii) *Q has Kazhdan's property (T);*
  (iii) *Q contains an isomorphic copy of C.*

If the G_n are non-elementary word hyperbolic groups, then claim (iii) *above can be replaced by*

(iii)′ *Q is periodic.*

This result immediately implies the validity of templates **FP** and **NA**$_{fg}$. Indeed, if G_n are the hypothesized groups of template **FP**, the preceding theorem furnishes us with a group Q that, for each $n \in \mathbb{N}$, is a quotient of G_n and hence cannot act without a fixed point on any $X \in \mathcal{X}_n$. Now let G_n be the hypothesized groups of template **NA**$_{fg}$. They are not assumed to be relatively hyperbolic. We consider groups $A_n := G_n * G_n * G_n$, which also cannot act non-trivially on any $X \in \mathcal{X}_n$. The group A_n is non-elementary and relatively hyperbolic as a free product of three non-trivial groups. Therefore, Theorem 1.6 can be applied to the sequence of groups A_n, providing a group Q_1 which, as a quotient of A_n, cannot act non-trivially on any $X \in \mathcal{X}_n$ for any $n \in \mathbb{N}$.

Following the above strategy to prove Theorem 1.1, we first represent $\mathcal{X}_{ac}$ as a countable union $\mathcal{X}_{ac} = \cup_{n,p}\mathcal{X}_{n,p}$, where, for each prime number p, the class $\mathcal{X}_{n,p}$ consists of all mod-p acyclic spaces of dimension n. Then, in Section 3, we construct the groups required by template **FP**, proving the following result.

Theorem 1.7. *For each $n \in \mathbb{N}$ and every prime p, there exists a non-elementary word hyperbolic group $G_{n,p}$ such that any action of $G_{n,p}$ by homeomorphisms on any space $X \in \mathcal{X}_{n,p}$ has the property that the global fixed point set is mod-p acyclic (and in particular non-empty).*

The mod-p acyclicity of the fixed point for the action of $G_{n,p}$ on the space X is a consequence of the following (n,p)-*generation property*: there is a generating set S of $G_{n,p}$ of cardinality $n+2$ such that any proper subset of S generates a finite p-subgroup.

For certain small values of the parameters examples of non-elementary word hyperbolic groups with the (n,p)-generation property were already known (e.g., when $n = 1$ and $p = 2$ they arise as reflection groups of the hyperbolic plane with a triangle as a fundamental domain). Our construction works for arbitrary n and p. For large n it provides the first examples of non-elementary word hyperbolic groups possessing the (n,p)-generation property.

We construct the groups $G_{n,p}$ as fundamental groups of certain simplices of groups all of whose local groups are finite p-groups. We use ideas related to simplicial non-positive curvature, developed by Januszkiewicz and Świątkowski in [10], to show that these groups are non-elementary word hyperbolic. The required fixed point property is obtained using Smith theory and a homological version of Helly's theorem.

Thus, to complete the proof of Theorem 1.1 and Corollary 1.2, it remains to prove Theorems 1.6 and 1.7. This will be done in Sections 2 and 3, respectively.

The validity of template $\mathbf{NA}_{fp}$ will be established in Section 4; it relies on the Higman Embedding Theorem. Also contained in Section 4 is Lemma 4.5, which establishes a triviality property for actions on manifolds. This is used both to provide input to the template $\mathbf{NA}_{fp}$ and to prove Proposition 1.4.

2. Relatively hyperbolic groups and their common quotients

Our purpose in this section is to provide the background we need concerning relatively hyperbolic groups and their quotients. This will allow us to prove Propositions 2.6 and 2.8 below, which immediately imply the assertion of Theorem 1.6. We adopt the combinatorial approach to relative hyperbolicity that was developed by Osin in [19].

Assume that G is a group, $\{H_\lambda\}_{\lambda \in \Lambda}$ is a fixed collection of proper subgroups of G (called *peripheral subgroups*), and A is a subset of G. The subset A is called a *relative generating set of G* with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$ if G is generated by $A \cup \bigcup_{\lambda \in \Lambda} H_\lambda$. In this case G is a quotient of the free product

$$F = (*_{\lambda \in \Lambda} H_\lambda) * F(A),$$

where $F(A)$ is the free group with basis A. Let $\mathcal{R}$ be a subset of F such that the kernel of the natural epimorphism $F \twoheadrightarrow G$ is the normal closure of $\mathcal{R}$ in the group F. In this case we will say that G has the *relative presentation*

(1) $$\langle A, \{H_\lambda\}_{\lambda \in \Lambda} \parallel R = 1, R \in \mathcal{R} \rangle.$$

If the sets A and $\mathcal{R}$ are finite, the relative presentation (1) is said to be *finite*.

Set $\mathcal{H} = \bigsqcup_{\lambda \in \Lambda}(H_\lambda \setminus \{1\})$. A finite relative presentation (1) is said to satisfy a *linear relative isoperimetric inequality* if there exists $C > 0$ such that for every word w in the alphabet $A \cup \mathcal{H}$ (for convenience, we will further assume that $A^{-1} = A$) representing the identity in the group G, one has

$$w \stackrel{F}{=} \prod_{i=1}^{k} f_i^{-1} R_i^{\pm 1} f_i,$$

with equality in the group F, where $R_i \in \mathcal{R}$, $f_i \in F$, for $i = 1, \ldots, k$, and $k \leq C\|w\|$, where $\|w\|$ is the length of the word w.

Definition 2.1. [19] The group G is said to be *relatively hyperbolic* if there is a collection $\{H_\lambda\}_{\lambda \in \Lambda}$ of proper peripheral subgroups of G such that G admits a finite relative presentation (1) satisfying a linear relative isoperimetric inequality.

This definition is independent of the choice of the finite generating set A and the finite set $\mathcal{R}$ in (1) (see [19]).

The definition immediately implies the following basic facts (see [19]):

Remark 2.2. (a) Let $\{H_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary family of groups. Then the free product $G = *_{\lambda \in \Lambda} H_\lambda$ will be hyperbolic relative to $\{H_\lambda\}_{\lambda \in \Lambda}$.

(b) Any word hyperbolic group (in the sense of Gromov) is hyperbolic relative to the family $\{\{1\}\}$, where $\{1\}$ denotes the trivial subgroup.

The following result is our main tool for constructing common quotients of countable families of relatively hyperbolic groups. Recall that a group G is said to be *non-elementary* if it does not contain a cyclic subgroup of finite index.

Theorem 2.3. [1, Thm. 1.4] *Any two finitely generated non-elementary relatively hyperbolic groups G_1, G_2 have a common non-elementary relatively hyperbolic quotient H.*

Consider a sequence of groups $(G_n)_{n \in \mathbb{N}}$ such that $G_i = G_1/K_i$, $i = 2, 3, \ldots$, for some $K_i \lhd G_1$ and $K_i \leq K_{i+1}$ for all $i \in \mathbb{N}$, $i \geq 2$. The *direct limit* of the sequence $(G_n)_{n \in \mathbb{N}}$ is, by definition, the group $G_\infty = G_1/K_\infty$ where $K_\infty = \bigcup_{n=2}^{\infty} K_n$.

Remark 2.4. If G_1 is finitely generated and G_n is infinite for every $n \in \mathbb{N}$, then G_∞ is also infinite.

Indeed, suppose that G_∞ is finite, i.e., $|G_1 : K_\infty| < \infty$. Then K_∞ is finitely generated as a subgroup of G_1, hence there exists $m \in \mathbb{N}$ such that $K_\infty = K_m$, and $G_\infty = G_m$ is infinite; this is a contradiction.

Remark 2.5. Any infinite finitely generated group G contains a normal subgroup N that is maximal with respect to the property $|G : N| = \infty$.

Indeed, let $\mathcal{N}$ be the set of all normal subgroups of infinite index in G ordered by inclusion. Consider a chain $(M_i)_{i \in I}$ in $\mathcal{N}$. Set $M = \cup_{i \in I} M_i$; then, evidently, $M \lhd G$. Now, if M had finite index in G, then it would also be finitely generated. Hence, by the definition of a chain, there would exist $j \in I$ such that $M = M_j$, which would contradict the assumption $|G : M_j| = \infty$. Therefore $M \in \mathcal{N}$ is an upper bound for the chain $(M_i)_{i \in I}$. Consequently, one can apply Zorn's Lemma to achieve the required maximal element of $\mathcal{N}$.

Proposition 2.6. *Let $\{G_i\}_{i \in \mathbb{N}}$ be a countable collection of finitely generated non-elementary relatively hyperbolic groups and let C be an arbitrary countable group. Then there exists a finitely generated group Q such that*

  (i) *Q is a quotient of G_i for every $i \in \mathbb{N}$;*
  (ii) *Q is a simple group;*
  (iii) *Q has Kazhdan's property (T);*
  (iv) *Q contains an isomorphic copy of C.*

Proof. First, embed C into an infinite finitely generated simple group S (see [14, Ch. IV, Thm. 3.5]). Let S' be a copy of S. Then the group $K = S * S'$ will be non-elementary and hyperbolic relative to the family consisting of two subgroups $\{S, S'\}$. Take G_0 to be an infinite word hyperbolic group that has property (T). Then G_0 is non-elementary

and relatively hyperbolic by Remark 2.2, hence we can use Theorem 2.3 to find a non-elementary relatively hyperbolic group $G(0)$ that is a common quotient of K and G_0 (in particular, $G(0)$ will also be finitely generated). Now, apply Theorem 2.3 to the groups $G(0)$ and G_1 to obtain their common non-elementary relatively hyperbolic quotient $G(1)$. Similarly, define $G(i)$ to be such a quotient for the groups $G(i-1)$ and G_i, $i = 2, 3, \ldots$. Let $G(\infty)$ be the direct limit of the sequence $\big(G(i)\big)_{i=0}^{\infty}$.

The group $G(\infty)$ is finitely generated (as a quotient of $G(0)$) and infinite (by Remark 2.4), therefore, by Remark 2.5, there exists a normal subgroup $N \lhd G(\infty)$ that is maximal with respect to the property $|G(\infty) : N| = \infty$. Set $Q = G(\infty)/N$. Then Q is an infinite group which has no non-trivial normal subgroups of infinite index. Being a quotient of $G(0)$, makes Q a quotient of $K = S * S'$, therefore it can not have any proper subgroups of finite index. Thus, Q is simple. Since the homomorphism $\varphi : S * S' \to Q$ has a non-trivial image, it must be injective on either S or S'. Therefore Q will contain an isomorphically embedded copy of S, and, consequently, of C.

The property (i) for Q follows from the construction. The property (iii) holds because Q is a quotient of G_0 and since Kazhdan's property (T) is stable under passing to quotients. $\qquad\square$

In the case when one has a collection of word hyperbolic groups (in the usual, non-relative, sense), one can obtain common quotients with different properties by using Ol'shanskii's theory of small cancellation over hyperbolic groups. For example, it is shown in [16] that if g is an element of infinite order in a non-elementary word hyperbolic group G, then there exists a number $n > 0$ such that the quotient of G by the normal closure of g^n is again a non-elementary word hyperbolic group. By harnessing this result to the procedure for constructing direct limits used in the proof of Proposition 2.6, we obtain the following statement, first proved by Osin:

Theorem 2.7. [20, Thm. 4.4] *There exists an infinite periodic group O, generated by two elements, such that for every non-elementary word hyperbolic group H there is an epimorphism $\rho : H \twoheadrightarrow O$.*

Proposition 2.8. *There exists an infinite finitely generated group Q such that*
   (a) *Q is a quotient of every non-elementary word hyperbolic group;*
   (b) *Q is a simple group;*
   (c) *Q has Kazhdan's property (T);*
   (d) *Q is periodic.*

Proof. Let O be the group given by Theorem 2.7. Since O is finitely generated, it has a normal subgroup $N \lhd O$ maximal with respect to the property $|O : N| = \infty$ (see Remark 2.5). Set $Q = O/N$. Then Q has no non-trivial normal subgroups of infinite index and is a quotient of every non-elementary word hyperbolic group; thus Q satisfies (a). In addition, Q is periodic since it is a quotient of O.

Observe that for an arbitrary integer $k \geq 2$ there exists a non-elementary word hyperbolic group $H = H(k)$ which does not contain any normal subgroups of index k

(for instance, one can take H to be the free product of two sufficiently large finite simple groups, e.g., $H = Alt(k+3) * Alt(k+3)$). Therefore the group Q, as a quotient of H, does not contain any normal subgroups of index k, for every $k \geq 2$, hence it is simple. It satisfies Kazhdan's property (T) because there are non-elementary word hyperbolic groups with (T) and property (T) is inherited by quotients. $\square$

Remark 2.9. The method that we used to obtain simple quotients in the proofs of Propositions 2.6 and 2.8 was highly non-constructive as it relied on the existence of a maximal normal subgroup of infinite index provided by Zorn's lemma. However, one can attain simplicity of the direct limit in a much more explicit manner, by imposing additional relations at each step. For word hyperbolic groups this was done in [15, Cor. 2]. The latter method for constructing direct limits of word hyperbolic groups was originally described by Ol'shanskii in [16]; it provides significant control over the resulting limit group. This control allows one to ensure that the group Q enjoys many properties in addition to the ones listed in the claim of Proposition 2.8. For example, in Proposition 2.8 one can add that Q has solvable word and conjugacy problems.

3. Simplices of finite p-groups with non-elementary word hyperbolic direct limits

Theorem 1.7 is an immediate consequence of the following two results, whose proof is the object of this section.

Theorem 3.1. *For every prime number p and integer $n \geq 1$ there is a non-elementary word hyperbolic group G generated by a set S of cardinality $n+2$ such that the subgroup of G generated by each proper subset of S is a finite p-group.*

Theorem 3.2. *Let p be a prime number. Suppose that a group G has a generating set S of cardinality $n+2$, such that the subgroup generated by each proper subset of S is a finite p-group. Then for any action of G on a Hausdorff mod-p acyclic space X of covering dimension less than or equal to n, the global fixed point set is mod-p acyclic.*

We prove Theorem 3.1 by constructing each of the desired groups as the fundamental group (equivalently, the direct limit) of a certain $(n+1)$-dimensional simplex of finite p-groups. The construction of the local groups in each simplex of groups, for fixed p, proceeds by induction on n. Provided that $m \leq n$, the groups that are assigned to each codimension m face of the $(n+1)$-simplex will depend only on m, up to isomorphism. The codimension zero face, i.e., the whole $(n+1)$-simplex itself, will be assigned the trivial group 1, and each codimension one simplex will be assigned a cyclic group of order p. As part of the inductive step, we will show that the fundamental group of the constructed $(n+1)$-simplex of groups maps onto a p-group in such a way that each local group maps injectively. This quotient p-group will be the group used as each vertex group in the $(n+2)$-simplex of groups.

The idea that drives our construction consists of requiring and exploiting existence of certain retraction homomorphisms between the local groups of the complexes of groups

involved. We develop this approach in Subsections 3.2–3.4 below, after recalling in Subsection 3.1 some basic notions and facts related to complexes of groups.

Each simplex of groups that we construct will be developable. Associated to any developable n-simplex of groups $\mathbf{G}$, there is a simplicial complex X on which the fundamental group G of $\mathbf{G}$ acts with an n-simplex as strict fundamental domain. If the local groups of $\mathbf{G}$ are all finite, the corresponding action is proper. Thus we may show that the group G is word hyperbolic by showing that the associated simplicial complex X is Gromov hyperbolic. We show that X is indeed hyperbolic by verifying that it satisfies a combinatorial criterion for the hyperbolicity of a simplicial complex related to the idea of simplicial non-positive curvature developed in [10]. More precisely, we show that X is 8-systolic, and hence hyperbolic. This is the content of Subsection 3.5.

From the perspective of the subject of simplicial non-positive curvature, Subsections 3.1-3.5 may be viewed as providing an alternative to the construction from [10] of numerous examples of k-systolic groups and spaces, for arbitrary k and in arbitrary dimension. The resulting groups are different from those obtained in [10].

From the algebraic perspective, this construction provides new operations of product type for groups, the so called *n-retra-products*, which interpolate between the direct product and the free product. These operations can be further generalized in the spirit of graph products. We think the groups obtained this way deserve to be studied. Such groups fall in the class of systolic, or even 8-systolic groups, and thus share various exotic properties of the latter, as established in [10, 11, 17, 18]. In a future work we plan to show that n-retra-products of finite groups, for sufficiently large n, are residually finite.

The last subsection of this section, Subsection 3.6, contains the proof of Theorem 3.2. This proof uses a result from Smith theory concerning mod-p cohomology of the fixed point set of a finite p-group action. It also uses a homological version of Helly's theorem for mod-p acyclic subsets.

Remark 3.3. It is because we need to apply Smith theory that the groups previously constructed in [10] are unsuitable for our purposes. The groups constructed in [10] include fundamental groups of simplices of *finite* groups which are non-elementary word hyperbolic, but the finite groups occurring in [10] are not of prime power order. One can show that the fundamental group of any $(n+1)$-simplex of finite groups cannot act without a global fixed point by isometries on any complete CAT(0) space of covering dimension at most n. Indeed, the Helly-type argument goes through almost unchanged, while the fact that the fixed point set for a finite group of isometries of a complete CAT(0) space is contractible replaces the appeal to Smith theory. This argument originates in unpublished work of Farb, and was extended by Bridson (also unpublished). A special case appears in [2].

3.1. **Strict complexes of groups.** We recall some basic notions and facts related to strict complexes of groups. The main reference is Bridson and Haefliger [4], where these objects are called simple complexes of groups.

A simplicial complex K gives rise to two categories: the category $\mathcal{Q}_K$ of non-empty simplices of K with inclusions as morphisms, and the extended category $\mathcal{Q}_K^+$ of simplices of K including the empty set $\emptyset$ as the unique (-1)–simplex. In addition to the morphisms from $\mathcal{Q}_K$, the category $\mathcal{Q}_K^+$ has one morphism from $\emptyset$ to σ for each nonempty simplex σ of K. A *strict complex of groups* $\mathbf{G}$ consists of a simplicial complex $|\mathbf{G}|$ (called *the underlying complex of* $\mathbf{G}$), together with a contravariant functor $\mathbf{G}$ from $\mathcal{Q}_{|\mathbf{G}|}$ to the category of groups and embeddings. A strict complex of groups is *developable* if the functor $\mathbf{G}$ extends to a contravariant functor $\mathbf{G}^+$ from the category $\mathcal{Q}_{|\mathbf{G}|}^+$ to the category of groups and embeddings. Given an extension $\mathbf{G}^+$ of $\mathbf{G}$, we will denote by G the group $\mathbf{G}^+(\emptyset)$. For simplices $\tau \subset \sigma$ (allowing $\tau = \emptyset$), we will view the group $\mathbf{G}(\sigma)$ as subgroup in the group $\mathbf{G}(\tau)$. We will be interested only in extensions that are *surjective*, i.e. such that the group $G = \mathbf{G}^+(\emptyset)$ is generated by the union of its subgroups $\mathbf{G}(\sigma)$ with $\sigma \neq \emptyset$.

We call any surjective extension $\mathbf{G}^+$ of $\mathbf{G}$ an *extended complex of groups*. We view the collection of all possible surjective extensions of $\mathbf{G}$ to $\mathcal{Q}_{|\mathbf{G}|}^+$ also as a category, which we denote by $\mathrm{Ext}_\mathbf{G}$. We take as morphisms of $\mathrm{Ext}_\mathbf{G}$ the natural transformations from $\mathbf{G}^+$ to $\mathbf{G}^{+\prime}$ which extend the identical natural transformation of $\mathbf{G}$. (Note that, given extensions $\mathbf{G}^+$ and $\mathbf{G}^{+\prime}$, there may be no morphism between them, and if there is one then, by surjectivity, it is unique; moreover, the homomorphism from $G = \mathbf{G}^+(\emptyset)$ to $G' = \mathbf{G}^{+\prime}(\emptyset)$ induced by a morphism is *not* required to be an embedding, although it is required to be a group homomorphism.) If $\mathbf{G}$ is developable, the category $\mathrm{Ext}_\mathbf{G}$ has an initial object $\mathbf{G}_{dir}^+$, in which the group $\mathbf{G}_{dir}^+(\emptyset)$ is just the direct limit of the functor $\mathbf{G}$ (for brevity, we often denote this direct limit group $\widetilde{G}$). Thus for any extension $\mathbf{G}^+$ of $\mathbf{G}$, there is a unique group homomorphism from $\widetilde{G}$ to G extending the identity map on each $\mathbf{G}(\sigma)$ for $\sigma \in \mathcal{Q}_{|\mathbf{G}|}$. In the cases that will be considered below, the simplicial complex $|\mathbf{G}|$ is simply connected, which implies that $\widetilde{G}$ coincides with what is known as the fundamental group of the complex of groups $\mathbf{G}$. (In fact, $|\mathbf{G}|$ is contractible in the cases considered below.)

For an extended complex of groups $\mathbf{G}^+$ we consider a space $d\mathbf{G}^+$ with an action of $G = \mathbf{G}^+(\emptyset)$, the *development* of $\mathbf{G}^+$, given by

$$d\mathbf{G}^+ = |\mathbf{G}| \times G/_\sim,$$

where the equivalence relation $\sim$ is given by $(p, g) \sim (q, h)$ iff $p = q$ and there exists $\sigma \in \mathcal{Q}_{|\mathbf{G}|}$ so that $p \in \sigma$ and $g^{-1}h \in \mathbf{G}(\sigma)$. It suffices to take σ to be the minimal simplex containing p. The G-action is given by $g[p, h] = [p, gh]$. The quotient by the action of G is (canonically isomorphic with) $|\mathbf{G}|$, and the subcomplex

$$[|\mathbf{G}|, 1] = \{[(p, 1)] : p \in |\mathbf{G}|\},$$

(where $[(p, 1)]$ is the equivalence class of $(p, 1)$ under $\sim$) is a strict fundamental domain for the action (in the sense that every G-orbit intersects $[|\mathbf{G}|, 1]$ in exactly one point). The space $d\mathbf{G}^+$ is a multi-simplicial complex, and the (pointwise and setwise)

stabilizer of the simplex $[\sigma, g]$ is the subgroup $g\mathbf{G}(\sigma)g^{-1}$. In the cases considered below, developments will be true simplicial complexes.

A *morphism* φ from a strict complex of groups $\mathbf{G}$ to a group H is a compatible collection of homomorphisms $\varphi_\sigma : \mathbf{G}(\sigma) \to H, \sigma \in \mathcal{Q}_{|\mathbf{G}|}$ (in general not necessarily injective). *Compatibility* means that we have equalities $\varphi_\sigma = \varphi_\tau \circ i_{\sigma\tau}$ for any $\tau \subset \sigma$, where $i_{\sigma\tau}$ is the inclusion of $\mathbf{G}(\sigma)$ in $\mathbf{G}(\tau)$. For example, a collection of inclusions $\mathbf{G}(\sigma) \to \mathbf{G}^+(\emptyset)$ is a morphism $\mathbf{G} \to \mathbf{G}^+(\emptyset)$. A morphism $\varphi : \mathbf{G} \to H$ is *locally injective* if all the homomorphisms φ_σ are injective.

Suppose we are given an action of a group H on a simplicial complex X, by simplicial automorphisms, and suppose this action is *without inversions*, i.e., if $g \in H$ fixes a simplex of X, it also fixes all vertices of this simplex. Suppose also that the action has a strict fundamental domain D which is a subcomplex of X. Clearly, D is then isomorphic to the quotient complex $H\backslash X$. Such an action determines the *extended associated complex of groups* $\mathbf{G}^+$, with the underlying complex $|\mathbf{G}| = D$, with local groups $\mathbf{G}(\sigma) := Stab(\sigma, H)$ for $\sigma \subset D$, and with $\mathbf{G}^+(\emptyset) := H$. The morphisms in $\mathbf{G}$ are the natural inclusions. It turns out that in this situation the development $d\mathbf{G}^+$ is H-equivariantly isomorphic with X.

3.2. Higher retractibility.

Now we pass to a less standard part of the exposition. We begin by describing a class of simplicial complexes, called *blocks*, that will serve as the underlying complexes of the complexes of groups involved in our construction. Then we discuss various requirements on the corresponding complexes of groups. Some part of this material is parallel to that in Sections 4 and 5 of [9], where retractibility and extra retractibility stand for what we call in this paper 1-retractibility and 2-retractibility, respectively.

Definition 3.4 (Block). A simplicial complex K of dimension n is a *chamber complex* if each of its simplices is a face of an n-simplex of K. Top dimensional simplices are then called *chambers* of K. A chamber complex K is *gallery connected* if each pair of its chambers is connected by a sequence of chambers in which any two consecutive chambers share a face of codimension 1. A chamber complex is *normal* if it is gallery connected and all of its links (which are also chamber complexes) are gallery connected. The *boundary* of a chamber complex K, denoted ∂K, is the subcomplex of K consisting of all those faces of codimension 1 that are contained in precisely one chamber. A *block* is a normal chamber complex with nonempty boundary. The *sides* of a block B are the faces of codimension 1 contained in ∂B. We denote the set of all sides of B by $\mathcal{S}_B$.

Note that links B_σ of a block at faces $\sigma \subset \partial B$ are also blocks, and that $\partial(B_\sigma) = (\partial B)_\sigma$.

Definition 3.5 (Normal block of groups). A *normal block of groups* over a block B is a strict complex of groups $\mathbf{G}$ with $|\mathbf{G}| = B$ satisfying the following two conditions:

  (1) $\mathbf{G}$ is *boundary supported*, i.e. $\mathbf{G}(\sigma) = 1$ for each σ not contained in ∂B;

   (2) $\mathbf{G}$ is *locally $\mathcal{S}$-surjective*, i.e., every group $\mathbf{G}(\sigma)$ is generated by the union $\bigcup\{\mathbf{G}(s) : s \in \mathcal{S}_B, \sigma \subset s\}$, where we use the convention that the empty set generates the trivial group 1.

An *extended normal block of groups* is an extension $\mathbf{G}^+$ of a normal block of groups $\mathbf{G}$ such that the associated morphism $\varphi : \mathbf{G} \to \mathbf{G}(\emptyset)$ is *$\mathcal{S}$-surjective*, i.e. $\mathbf{G}(\emptyset)$ is generated by the union $\bigcup\{\mathbf{G}(s) : s \in \mathcal{S}_B\}$. (To simplify notation, we write $\mathbf{G}(\sigma)$ instead of $\mathbf{G}^+(\sigma)$ to denote the corresponding groups of $\mathbf{G}^+$.)

Given an extended normal block of groups $\mathbf{G}^+$ over B, its development $d\mathbf{G}^+$ is tesselated by copies of B. More precisely, $d\mathbf{G}^+$ is the union of the subcomplexes of the form $[B, g]$, with $g \in \mathbf{G}(\emptyset)$, which do not intersect each other except at their boundaries, and which we view as tiles of the tesselation. Moreover, the action of $\mathbf{G}(\emptyset)$ on $d\mathbf{G}^+$ is simply transitive on these tiles. By $\mathcal{S}$-surjectivity of φ, $d\mathbf{G}^+$ is a normal chamber complex. If $|\mathbf{G}(s)| > 1$ for all $s \in \mathcal{S}_B$ then the chamber complex $d\mathbf{G}^+$ has empty boundary. If B is a pseudo-manifold and $|\mathbf{G}(s)| \leq 2$ for all $s \in \mathcal{S}_B$, then $d\mathbf{G}^+$ is a pseudo-manifold.

Definition 3.6 (1-retractibility). An extended normal block of groups $\mathbf{G}^+$ is *1-retractible* if for every $\sigma \subset |\mathbf{G}|$ there is a homomorphism $r_\sigma : \mathbf{G}(\emptyset) \to \mathbf{G}(\sigma)$ such that $r_\sigma|_{\mathbf{G}(s)} = id_{\mathbf{G}(s)}$ for $s \in \mathcal{S}_{|\mathbf{G}|}, s \supset \sigma$, and $r_\sigma|_{\mathbf{G}(s)} = 1$ otherwise.

In the next two lemmas we present properties that immediately follow from 1-retractibility. We omit the straightforward proofs.

Lemma 3.7.
   (1) *The homomorphisms r_σ, if they exist, are unique.*
   (2) *Let $\varphi_\sigma : \mathbf{G}(\sigma) \to \mathbf{G}(\emptyset)$ be the homomorphisms of the morphism $\varphi : \mathbf{G} \to \mathbf{G}(\emptyset)$. Then for each σ we have $r_\sigma \varphi_\sigma = id_{\mathbf{G}(\sigma)}$. Thus r_σ is a retraction onto the subgroup $\mathbf{G}(\sigma) < \mathbf{G}(\emptyset)$.*
   (3) *The inclusion homomorphisms $\varphi_{\tau\sigma} : \mathbf{G}(\tau) \to \mathbf{G}(\sigma)$, for $\sigma \subset \tau$, occurring as the structure homomorphisms of $\mathbf{G}$, satisfy $\varphi_{\tau\sigma} = r_\sigma \varphi_\tau$.*

Motivated by property (3) above, we define homomorphisms $r_{\rho\tau} : \mathbf{G}(\rho) \to \mathbf{G}(\tau)$, for any simplices ρ, τ of $|\mathbf{G}|$, including $\emptyset$, by putting $r_{\rho\tau} := r_\tau \varphi_\rho$.

Lemma 3.8. *Each of the homomorphisms $r_{\rho\tau}$ is uniquely determined by the following two requirements:*
   (1) $r_{\rho\tau}|_{\mathbf{G}(s)} = id_{\mathbf{G}(s)}$ *for* $s \in \mathcal{S}_{|\mathbf{G}|}, s \supset \rho, s \supset \tau$;
   (2) $r_{\rho\tau}|_{\mathbf{G}(s)} = 1$ *otherwise (i.e. for $s \in \mathcal{S}_{|\mathbf{G}|}, s \supset \rho, s$ not containing τ).*
In particular, we have $r_{\sigma\emptyset} = \varphi_\sigma$, $r_{\emptyset\sigma} = r_\sigma$, and $r_{\tau\sigma} = \varphi_{\tau\sigma}$ whenever $\tau \supset \sigma$. Moreover, if $\tau \supset \sigma$ then $r_{\tau\sigma}$ is a retraction (left inverse) for the inclusion $\varphi_{\tau\sigma}$.

To define higher retractibility properties for an extended normal block of groups $\mathbf{G}^+$ we need first to introduce certain new blocks of groups called *unfoldings* of $\mathbf{G}^+$ at the boundary simplices $\sigma \subset \partial|\mathbf{G}|$.

Definition 3.9 (Unfolding of $\mathbf{G}^+$ at σ)**.** Let $\mathbf{G}^+$ be a 1-retractible extended normal block of groups. Let $\sigma \subset \partial|\mathbf{G}|$ be a simplex, and denote by $d_\sigma\mathbf{G}(\emptyset)$ the kernel of the retraction homomorphism $r_\sigma : \mathbf{G}(\emptyset) \to \mathbf{G}(\sigma)$. The *unfolding* of $\mathbf{G}^+$ at σ, denoted $d_\sigma\mathbf{G}$, is the complex of groups associated to the action of the group $d_\sigma\mathbf{G}(\emptyset)$ on the development $d\mathbf{G}^+$. The *extended unfolding* $d_\sigma\mathbf{G}^+$ is the same complex of groups equipped with the canonical morphism to the group $d_\sigma\mathbf{G}(\emptyset)$.

Define a subcomplex $d_\sigma|\mathbf{G}| \subset d\mathbf{G}^+$ by $d_\sigma|\mathbf{G}| := \bigcup\{[B, g] : g \in \mathbf{G}(\sigma)\}$. We will show that the above defined unfolding $d_\sigma\mathbf{G}^+$ is an extended normal block of groups over $d_\sigma|\mathbf{G}|$. This will be done in a series of lemmas, in which we describe the structure of $d_\sigma|\mathbf{G}|$ and $d_\sigma\mathbf{G}^+$ in detail.

Lemma 3.10. $d_\sigma|\mathbf{G}|$ *is a strict fundamental domain for the action of the group* $d_\sigma\mathbf{G}(\emptyset)$ *on the development* $d\mathbf{G}^+$. *In particular,* $|d_\sigma\mathbf{G}| = d_\sigma|\mathbf{G}|$.

Proof. We need to show that the restriction to $d_\sigma|\mathbf{G}|$ of the quotient map $q_\sigma : d\mathbf{G}^+ \to d_\sigma\mathbf{G}(\emptyset)\backslash d\mathbf{G}^+$ is a bijection. This follows by observing that the map $j_\sigma : d_\sigma\mathbf{G}(\emptyset)\backslash d\mathbf{G}^+ \to d_\sigma|\mathbf{G}|$ defined by $j_\sigma(d_\sigma\mathbf{G}(\emptyset) \cdot [p, g]) = [p, r_\sigma(g)]$ is the inverse of $q_\sigma|_{d_\sigma|\mathbf{G}|}$. $\square$

To proceed with describing $d_\sigma|\mathbf{G}|$, we need to define links for blocks of groups. This notion will also be useful in our later considerations.

Definition 3.11 (Link of a block of groups)**.** Let $\mathbf{G}$ be a normal block of groups and let σ be a simplex of $|\mathbf{G}|$. The *link* of $\mathbf{G}^+$ at σ is an extended normal block of groups $\mathbf{G}_\sigma^+$ over the link $|\mathbf{G}|_\sigma$ given by $\mathbf{G}_\sigma^+(\tau) := \mathbf{G}(\tau * \sigma)$ for all $\tau \subset |\mathbf{G}|_\sigma$, including the empty set $\emptyset$ (with the convention that $\emptyset * \sigma = \sigma$).

We skip the straightforward argument for showing that the above defined extended complex of groups is an extended normal block of groups.

The next lemma describes the links of the complex $d_\sigma|\mathbf{G}|$. We omit its straightforward proof. In this lemma, and in the remaining part of this section, we will denote by $\sigma - \tau$ the face of σ spanned by the vertices of σ not contained in τ.

Lemma 3.12. *Let* $[\tau, g]$ *be a simplex of* $d_\sigma|\mathbf{G}|$, *where* $\tau \subset |\mathbf{G}|$ *and* $g \in \mathbf{G}(\sigma)$. *For any simplex* $\rho \subset |\mathbf{G}|_\tau$ *let* $d_\rho|\mathbf{G}_\tau|$ *be the strict fundamental domain for the action of the group* $d_\rho\mathbf{G}_\tau(\emptyset)$ *on the development* $d\mathbf{G}_\tau^+$. *Then the link of* $d_\sigma|\mathbf{G}|$ *at* $[\tau, g]$ *has one of the following two forms depending on* τ:
  (1) $(d_\sigma|\mathbf{G}|)_{[\tau,g]} \cong d_{\sigma-\tau}|\mathbf{G}_\tau|$ *if* σ *and* τ *span a simplex of* $|\mathbf{G}|$, *where we use convention that* $d_\emptyset|\mathbf{G}_\tau| = d\mathbf{G}_\tau^+$;
  (2) $(d_\sigma|\mathbf{G}|)_{[\tau,g]} \cong |\mathbf{G}|_\tau$ *otherwise.*

Lemma 3.12 easily implies the following corollary. The proof of part (1) uses induction on the dimension of B and $\mathcal{S}$-surjectivity of the extending morphism; we omit the details.

Corollary 3.13.

(1) $d_\sigma|\mathbf{G}|$ is a normal chamber complex.
(2) The boundary $\partial(d_\sigma|\mathbf{G}|)$ is the subcomplex of $d_\sigma|\mathbf{G}|$ consisting of the simplices of form $[\rho, g]$ for all $\rho \subset \partial|\mathbf{G}|$ not containing σ and for all $g \in \mathbf{G}(\sigma)$. In particular, the set of sides of $d_\sigma|\mathbf{G}|$ is the set

$$\mathcal{S}_{d_\sigma|\mathbf{G}|} = \{[s, g] : s \in \mathcal{S}_{|\mathbf{G}|}, s \text{ does not contain } \sigma, g \in \mathbf{G}(\sigma)\}.$$

For a subgroup $H < G$ and an element $g \in G$, we denote by H^g the conjugation gHg^{-1}. The next lemma describes the local groups of the unfolding $d_\sigma\mathbf{G}$.

Lemma 3.14. *Let $d_\sigma\mathbf{G}$ be the unfolding of $\mathbf{G}$ and let $[\tau, g]$ be a simplex of $d_\sigma|\mathbf{G}|$ (with $\tau \subset |\mathbf{G}|$ and $g \in \mathbf{G}_\sigma$). Then*

$$d_\sigma\mathbf{G}([\tau, g]) = [\ker(r_{\tau\sigma} : \mathbf{G}(\tau) \to \mathbf{G}(\sigma))]^g < d_\sigma\mathbf{G}(\emptyset),$$

and consequently

$$d_\sigma\mathbf{G}([\tau, g]) = g \cdot \langle \bigcup \mathbf{G}(s) : s \in \mathcal{S}_{|\mathbf{G}|}, s \supset \tau, s \text{ does not contain } \sigma \rangle \cdot g^{-1}.$$

In view of Corollary 3.13(2), Lemma 3.14 implies the following.

Corollary 3.15. *For simplices $[\tau, g]$ of $d_\sigma|\mathbf{G}|$ not contained in the boundary $\partial(d_\sigma|\mathbf{G}|)$ we have $d_\sigma\mathbf{G}([\tau, g]) = 1$.*

Another consequence of Lemma 3.14, which will be useful later, is the following.

Corollary 3.16. *Let $\mathbf{G}^+$ be an extended normal block of groups, and let σ be a simplex of $|\mathbf{G}|$. Then for any simplex $[\tau, g] \subset d_\sigma|\mathbf{G}|$, with $\tau \subset |\mathbf{G}|$ and $g \in \mathbf{G}(\sigma)$, we have $[d_\sigma\mathbf{G}]_{[\tau,g]} \cong d_{\sigma-\tau}\mathbf{G}_\tau^+$, where $\cong$ denotes an isomorphism of extended complexes of groups, and where $d_\emptyset\mathbf{G}_\tau^+$ denotes here the trivial strict complex of groups over $d\mathbf{G}_\tau^+$ (i.e. all of the local groups are trivial).*

As a consequence of the results above, from Lemma 3.10 to Corollary 3.15, we obtain the following.

Corollary 3.17. *Given an extended normal block of groups $\mathbf{G}^+$, each of its unfoldings $d_\sigma\mathbf{G}^+$ is an extended normal block of groups.*

We are now in a position to define recursively higher retractibilities.

Definition 3.18 (*n-retractibility*)**.** Let n be a natural number. An extended normal block of groups $\mathbf{G}^+$ is $(n+1)$-*retractible* if it is 1-retractible, and for every simplex $\sigma \subset \partial|\mathbf{G}|$ the unfolding $d_\sigma\mathbf{G}^+$ is n-retractible.

Example 3.19 (*n-retractible 1-simplex of groups*). Consider the extended complex of groups $\mathbf{G}^+$ with $|\mathbf{G}|$ equal to a 1-simplex, with the vertex groups $\mathbf{G}(v)$ cyclic of order two, and with $\mathbf{G}(\emptyset)$ dihedral of order $2k$, where generators of the vertex groups correspond to standard generators of the dihedral group. This complex of groups is clearly an extended normal block of groups. Moreover, it is n-retractible but not $(n+1)$-retractible in the case when $k = 2^n(2m + 1)$ for some m.

Remark 3.20.

(1) Note that an n-retractible extended normal block of groups is always k-retractible for each $k \leq n$.
(2) All links in a 1-retractible normal block of groups are 1-retractible.
(3) From Corollary 3.16 and the above remark (2) one can easily deduce using induction on n that if $\mathbf{G}^+$ is n-retractible and σ is a simplex of $|\mathbf{G}|$, then $\mathbf{G}_\sigma^+$ is also n-retractible.

3.3. **n-retractible extensions.** Our next objective is to establish results that give partial converses to property in Remark 3.20(3). These will allow us to pass up one dimension in our recursive construction of n-retractible simplices of groups, in the next subsection.

Proposition 3.21. *Let $\mathbf{G}$ be a normal block of groups. If for some natural number n all the links in $\mathbf{G}$ (as extended complexes of groups) are n-retractible then:*

(1) *$\mathbf{G}$ is developable, and*
(2) *the extension $\mathbf{G}_{dir}^+$ of $\mathbf{G}$ (in which $\mathbf{G}(\emptyset)$ coincides with the direct limit $\widetilde{G}$ of $\mathbf{G}$) is n-retractible.*

Proof. To prove part (1), we need to show that $\mathbf{G}$ admits a locally injective morphism $\psi : \mathbf{G} \to H$ to some group H. To find ψ, for each $\sigma \subset |\mathbf{G}|$ we construct a morphism $\bar{r}_\sigma : \mathbf{G} \to \mathbf{G}(\sigma)$ which is identical on the group $\mathbf{G}(\sigma)$ of $\mathbf{G}$. We then take as H the direct product $H = \oplus\{\mathbf{G}(\sigma) : \sigma \subset |\mathbf{G}|\}$ and as ψ the diagonal morphism $\psi = \oplus\{\bar{r}_\sigma : \sigma \subset |\mathbf{G}|\}$, which is then clearly locally injective.

Fix a simplex $\sigma \in |\mathbf{G}|$. To get a morphism $\bar{r}_\sigma$ as above, we will construct an appropriate compatible collection of homomorphisms $\bar{r}_{\eta\sigma} : \mathbf{G}(\eta) \to \mathbf{G}(\sigma)$, for all $\eta \subset |\mathbf{G}|$, such that $\bar{r}_{\sigma\sigma} = id_{\mathbf{G}(\sigma)}$. To do this, for any simplex $\rho \subset |\mathbf{G}|$ consider the set of sides $\mathcal{S}_\rho = \{s \in \mathcal{S}_{|\mathbf{G}|} : \rho \subset s\}$. Fix a simplex $\eta \subset |\mathbf{G}|$. If $\mathcal{S}_\eta \cap \mathcal{S}_\sigma = \emptyset$, put $\bar{r}_{\eta\sigma}$ to be the trivial homomorphism. Otherwise, consider the simplex $\tau = \cap\{s : s \in \mathcal{S}_\eta \cap \mathcal{S}_\sigma\}$. Clearly, we have then $\eta \subset \tau$ and $\sigma \subset \tau$. In particular, we have the inclusion homomorphism $i_{\tau\sigma} : \mathbf{G}(\tau) \to \mathbf{G}(\sigma)$, which is identical on the subgroups $\mathbf{G}(s) : s \in \mathcal{S}_\eta \cap \mathcal{S}_\sigma$.

Recall that we denote by $\tau - \eta$ the face of τ spanned by all vertices not contained in η. Since $\eta \subset \tau$, the groups $\mathbf{G}(\eta)$ and $\mathbf{G}(\tau)$ coincide with the link groups $\mathbf{G}_\eta^+(\emptyset)$ and $\mathbf{G}_\eta^+(\tau - \eta)$, respectively. Since the link $\mathbf{G}_\eta^+$ is 1-retractible, we have the retraction

$$r_{\tau-\eta} : \mathbf{G}_\eta^+(\emptyset) \to \mathbf{G}_\eta^+(\tau - \eta)$$

such that $r_{\tau-\eta}|_{\mathbf{G}_\eta^+(s)} = id_{\mathbf{G}_\eta^+(s)}$ for $s \in \mathcal{S}_{|\mathbf{G}|_\eta}, s \supset \tau - \eta$ and $r_{\tau-\eta}|_{\mathbf{G}_\eta^+(s)} = 1$ otherwise (cf. Definition 3.6).

Put $\bar{r}_{\eta\sigma} := i_{\tau\sigma} \circ r_{\tau-\eta}$. We claim that $\bar{r}_{\eta\sigma}$ satisfies the assertions (1) and (2) of Lemma 3.8, when substituted for $r_{\eta\sigma}$. This follows from the identification of the groups $\mathbf{G}(s), s \in \mathcal{S}_{|\mathbf{G}|}$ with the groups $\mathbf{G}_\eta^+(s-\eta)$, for $\eta \subset s$, and from the fact that $s-\eta \in \mathcal{S}_{|\mathbf{G}|_\eta}$ (because $\partial(|\mathbf{G}|_\eta) = (\partial|\mathbf{G}|)_\eta$).

Now, we need to check the compatibility condition $\bar{r}_{\eta_2\sigma} = \bar{r}_{\eta_1\sigma} \circ i_{\eta_2\eta_1}$ for all simplices $\eta_1 \subset \eta_2$ in $|\mathbf{G}|$. This follows from the coincidence of the maps on both sides of the equality on the generating set $\cup\{\mathbf{G}(s) : s \in \mathcal{S}_{|\mathbf{G}|}, s \supset \eta_2, \}$ of $\mathbf{G}(\eta_2)$. This coincidence is a fairly direct consequence of assertions (1) and (2) of Lemma 3.8, satisfied by the maps $\bar{r}_{\eta_1\sigma}$ and $\bar{r}_{\eta_2\sigma}$.

Finally, assertion (1) of Lemma 3.8 clearly implies that $\bar{r}_{\sigma\sigma} = id_{\mathbf{G}(\sigma)}$, which concludes the proof of developability of $\mathbf{G}$.

We now turn to proving part (2). To deal with the case $n = 1$ we need to construct the map $r_\sigma : \widetilde{G} \to \mathbf{G}(\sigma)$ as required in Definition 3.6, for any $\sigma \subset |\mathbf{G}|$.

Consider the maps $\bar{r}_{\eta\sigma} : \eta \subset |\mathbf{G}|$ constructed in the proof of part (1), and the morphism $\bar{r}_\sigma : \mathbf{G} \to \mathbf{G}(\sigma)$ given by these maps. Let $r_\sigma : \widetilde{G} \to \mathbf{G}(\sigma)$ be the homomorphism induced by this morphism. The requirements of Definition 3.6 for r_σ follow then easily from the assertions of Lemma 3.8 satisfied by the maps $\bar{r}_{\eta\sigma}$ (we skip the straightforward details). Thus 1-retractibility of links of $\mathbf{G}$ implies 1-retractibility of $\mathbf{G}^+_{dir}$.

Now suppose that $n > 1$. If links in $\mathbf{G}$ are n-retractible, it follows that links in $d_\sigma\mathbf{G}$ are $(n-1)$-retractible for all σ. By induction, it follows that the unfoldings $d_\sigma\mathbf{G}^+_{dir}$ are $(n-1)$-retractible, and so $\mathbf{G}^+_{dir}$ is n-retractible. $\qquad\square$

Example 3.22 (*n*-retractible 2-simplex of groups). Consider the triangle Coxeter group $W_{n,m}$ given by

$$W_{n,m} = \langle s_1, s_2, s_3 | s_i^2, (s_i s_j)^k \text{ for } i \neq j \rangle,$$

where $k = 2^n(2m + 1)$. The Coxeter complex for this group is a triangulation of the (hyperbolic) plane on which the group acts by simplicial automorphisms, simply transitively on 2-simplices. Let $\mathbf{G}^+$ be the extended 2-simplex of groups associated to this action. Links in $\mathbf{G}$ at vertices are then isomorphic to the 1-simplex of groups from Example 3.18. Thus, in view of Proposition 3.21 and Remark 3.20(3), $\mathbf{G}^+$ is n-retractible, but not $(n+1)$-retractible.

A more subtle way of getting n-retractible extensions is given in the following theorem, which will be used directly, as a recursive step, in our construction in Subsection 3.4.

Theorem 3.23. *Let $\mathbf{G}$ be a normal block of groups in which all links are n-retractible. Then there exists an extension $\mathbf{G}^+_{\min}$ of $\mathbf{G}$ that has the following properties.*

(1) *$\mathbf{G}^+_{\min}$ is the minimal n-retractible extension of $\mathbf{G}$ in the following sense: if $\mathbf{G}^+$ is any n-retractible extension of $\mathbf{G}$ then there is a unique morphism of extended complexes of groups $\mathbf{G}^+ \to \mathbf{G}^+_{\min}$ which extends the identity on $\mathbf{G}$ (i.e. a morphism in the category $\mathrm{Ext}_\mathbf{G}$).*

(2) *If all $\mathbf{G}(\sigma)$ are finite, so is $\mathbf{G}_{\min}(\emptyset)$.*

(3) *If all $\mathbf{G}(\sigma)$ are p-groups of bounded exponent, so is $\mathbf{G}_{\min}(\emptyset)$.*

(4) *If all $\mathbf{G}(\sigma)$ are soluble groups with soluble length $\leq d$ for some fixed d, then so is $\mathbf{G}_{\min}(\emptyset)$.*

Remark 3.24. Property 1 means that the extended complex of groups $\mathbf{G}^+_{\min}$ is the terminal object in the category of n-retractible extensions of $\mathbf{G}$. In particular, it is unique.

Proof. Let $\mathbf{G}_{dir}^+$ be the direct limit extension of $\mathbf{G}$, as described in Proposition 3.21. We recursively define iterated unfoldings of $\mathbf{G}_{dir}^+$. For each simplex σ_1 of $|\mathbf{G}|$, define $d_{\sigma_1}\mathbf{G}_{dir}^+$ as previously. Suppose that a complex of groups $d_{\sigma_1,\ldots,\sigma_k}\mathbf{G}_{dir}^+$ has already been defined. For each simplex σ_{k+1} of the underlying simplicial complex $|d_{\sigma_1,\ldots,\sigma_k}\mathbf{G}_{dir}^+|$, let $d_{\sigma_1,\ldots,\sigma_k,\sigma_{k+1}}\mathbf{G}_{dir}^+ = d_{\sigma_{k+1}}(d_{\sigma_1,\ldots,\sigma_k}\mathbf{G}_{dir}^+)$. Since $\mathbf{G}$ is n-retractible, this allows us to define extended complexes of groups $d_{\sigma_1,\ldots,\sigma_k}\mathbf{G}_{dir}^+$ for any $k \leq n$. Due to Corollary 3.17, all of these are extended normal blocks of groups.

Note that each of the groups $d_{\sigma_1,\ldots,\sigma_k}\mathbf{G}_{dir}(\emptyset)$, which we denote $\widetilde{G}_{\sigma_1,\ldots,\sigma_k}$, is a subgroup of the direct limit $\widetilde{G}$ of $\mathbf{G}$. Let N_k be the largest normal subgroup of $\widetilde{G}$ which is contained in every subgroup $\widetilde{G}_{\sigma_1,\ldots,\sigma_k}$, where $\sigma_1,\ldots,\sigma_k$ ranges over all *allowed* sequences of simplices (i.e., all sequences for which $d_{\sigma_1,\ldots,\sigma_k}\mathbf{G}_{dir}^+$ was defined above). If we set $\mathbf{G}_{\min}(\emptyset) = \widetilde{G}/N_n$, the resulting extension is clearly n-retractible. We need to show it is minimal.

Let $\mathbf{G}_0^+$ be any n-retractible extension of $\mathbf{G}$. Clearly, we have the canonical surjective homomorphism $h : \widetilde{G} \to \mathbf{G}_0(\emptyset)$, which allows us to express $\mathbf{G}_0(\emptyset)$ as the quotient $\widetilde{G}/\ker h$. To get the homomorphism $\mathbf{G}_0(\emptyset) \to \mathbf{G}_{\min}(\emptyset) = \widetilde{G}/N_n$ as required for minimality, we need to show that $\ker h < N_n$. By definition of N_n, it is thus sufficient to show that $\ker h < \widetilde{G}_{\sigma_1,\ldots,\sigma_n}$ for all allowed sequences $\sigma_1,\ldots,\sigma_n$.

To prove the latter, we will show by induction on k that $\ker h < \widetilde{G}_{\sigma_1,\ldots,\sigma_k}$, for all $1 \leq k \leq n$. For $k = 1$, we have retractions $\widetilde{G} \to \mathbf{G}(\sigma_1)$ and $\mathbf{G}_0(\emptyset) \to \mathbf{G}(\sigma_1)$ which commute with h, and such that $\widetilde{G}_{\sigma_1} = \ker[\widetilde{G} \to \mathbf{G}(\sigma_1)]$ and $d_{\sigma_1}\mathbf{G}_0(\emptyset) = \ker[\mathbf{G}_0(\emptyset) \to \mathbf{G}(\sigma_1)]$. Consequently, we get a canonical homomorphism $h_{\sigma_1} : \widetilde{G}_{\sigma_1} \to d_{\sigma_1}\mathbf{G}_0(\emptyset)$ with $\ker H_{\sigma_1} = \ker h$. In particular, $\ker h < \widetilde{G}_{\sigma_1}$.

To proceed, observe that (due to Lemma 3.14 applied recursively) the non-extended unfoldings $d_{\sigma_1,\ldots,\sigma_k}\mathbf{G}_{dir}$ and $d_{\sigma_1,\ldots,\sigma_k}\mathbf{G}_0$ coincide for all $1 \leq k \leq n$. Thus, for any $\sigma_2 \subset |d_{\sigma_1}\mathbf{G}_{dir}| = |d_{\sigma_1}\mathbf{G}_0|$ we can repeat the above argument and get the homomorphism $h_{\sigma_1,\sigma_2} : \widetilde{G}_{\sigma_1,\sigma_2} \to d_{\sigma_1\sigma_2}\mathbf{G}_0(\emptyset)$ with $\ker h_{\sigma_1,\sigma_2} = \ker h_{\sigma_1} = \ker h$. Repeating this argument, we finally get the homomorphisms $h_{\sigma_1,\ldots,\sigma_n} : \widetilde{G}_{\sigma_1,\ldots,\sigma_n} \to d_{\sigma_1,\ldots,\sigma_n}\mathbf{G}_0(\emptyset)$ with $\ker h_{\sigma_1,\ldots,\sigma_n} = \ker h$. It follows that $\ker h < \widetilde{G}_{\sigma_1,\ldots,\sigma_n}$. This proves statement 1.

To prove statement 2, recall that, by Definition 3.4, the block $|\mathbf{G}|$ is finite. If $\mathbf{G}(\sigma)$ is finite for each $\sigma \neq \emptyset$, then it follows (by applying recursively Lemmas 3.10 and 3.14) that each underlying simplicial complex $|d_{\sigma_1,\ldots,\sigma_k}\mathbf{G}_{dir}|$ is finite, and that for every non-empty simplex $\eta \subset |d_{\sigma_1,\ldots,\sigma_k}\mathbf{G}_{dir}|$ the group $d_{\sigma_1,\ldots,\sigma_k}\mathbf{G}_{dir}(\eta)$ is finite. Hence each $\widetilde{G}_{\sigma_1,\ldots,\sigma_k}$ has finite index in $\widetilde{G}$, and there are finitely many such groups for each $k \leq n$. If follows that the intersection of all such subgroups is a subgroup of finite index, and so each N_k for $k \leq n$ is a finite index normal subgroup of $\widetilde{G}$. This proves that $\mathbf{G}_{\min} = \widetilde{G}/N_n$ is a finite group as claimed in statement 2.

Before proving statements 3 and 4, we first claim that each N_k is equal to the intersection of the groups $\widetilde{G}_{\sigma_1,\ldots,\sigma_k}$. To see this, it is useful to change the indexing set. For

$\tau_1, \ldots, \tau_k$ a sequence of simplices of $d\mathbf{G}^+_{dir}$ of length at most n, define σ_1 to be the image of τ_1 in $|\mathbf{G}| = \widetilde{G}\backslash d\mathbf{G}^+_{dir}$. Assuming that $\sigma_1, \ldots, \sigma_{i-1}$ have already been defined for some i with $1 < i \leq k$, define σ_i to be the image of τ_i in $|d_{\sigma_1,\ldots,\sigma_{i-1}}\mathbf{G}^+_{dir}| = \widetilde{G}_{\sigma_1,\ldots,\sigma_{i-1}}\backslash d\mathbf{G}^+_{dir}$. Also define $d_{\tau_1\ldots,\tau_k}\mathbf{G}^+_{dir}$ to be equal to $d_{\sigma_1,\ldots,\sigma_k}\mathbf{G}^+_{dir}$, and define $\widetilde{G}_{\tau_1\ldots,\tau_k}$ to be equal to $\widetilde{G}_{\sigma_1,\ldots,\sigma_k}$.

If x is a point of $d\mathbf{G}^+_{dir}$ whose stabilizer is some subgroup $H < \widetilde{G}$, and if g is an element of $\widetilde{G}$, the stabilizer of the point $g.x$ is equal to the conjugation gHg^{-1}. This observation and induction show that for each $g \in \widetilde{G}$, for each $k \leq n$ and for each sequence $\tau_1, \ldots, \tau_k$ of simplices of $d\mathbf{G}^+_{dir}$, we have

$$\widetilde{G}_{g\tau_1,\ldots,g\tau_k} = g \cdot \widetilde{G}_{\tau_1,\ldots,\tau_k} \cdot g^{-1}.$$

Hence the intersection of the subgroups of the form $\widetilde{G}_{\tau_1,\ldots,\tau_k}$, for fixed $k \leq n$, is a normal subgroup of $\widetilde{G}$. It follows that this intersection is equal to N_k.

The above observation combined with the inclusions $\widetilde{G}_{\sigma_1,\ldots,\sigma_k} < \widetilde{G}_{\sigma_1,\ldots,\sigma_k,\sigma_{k+1}}$ implies that $N_{k+1} < N_k$. To prove statement 3, we will show by induction that for $0 \leq k < n$ the quotient groups N_k/N_{k+1} and $\widetilde{G}/N_{k+1}$ are p-groups of bounded exponent. Here we use convention that $N_0 = \widetilde{G}$.

For $k = 0$, we need to show that $\widetilde{G}/N_1$ is a p-group of bounded exponent. We know that N_1 is the intersection of the groups of form $\widetilde{G}_{\sigma_1}$, which are the kernels of retractions $\widetilde{G} \to \mathbf{G}(\sigma_1)$. Thus $\widetilde{G}/N_1$ embeds in the product of the groups $\mathbf{G}(\sigma_1)$. Since the latter groups are p-groups of bounded exponent, the assertion follows.

Now we suppose $\widetilde{G}/N_k$ is a p-group of bounded exponent and claim that the group N_k/N_{k+1} is too. To see that this is true, recall that the groups $\widetilde{G}_{\sigma_1,\ldots,\sigma_{k+1}}$ are the kernels of the retractions $\widetilde{G}_{\sigma_1,\ldots,\sigma_k} \to d_{\sigma_1\ldots\sigma_k}\mathbf{G}_{dir}(\sigma_{k+1})$. Note also that

$$N_{k+1} = \bigcap \widetilde{G}_{\sigma_1,\ldots,\sigma_{k+1}} = \bigcap [N_k \cap \widetilde{G}_{\sigma_1,\ldots,\sigma_{k+1}}],$$

and thus N_{k+1} is equal to the intersection of the kernels of the composed homomorphisms

$$N_k \xrightarrow{\text{incl}} \widetilde{G}_{\sigma_1,\ldots,\sigma_k} \xrightarrow{r} d_{\sigma_1,\ldots,\sigma_k}\mathbf{G}_{dir}(\sigma_{k+1}).$$

Now, each of the groups $d_{\sigma_1,\ldots,\sigma_k}\mathbf{G}_{dir}(\sigma_{k+1})$ canonically embeds in the quotient $\widetilde{G}/N_k$ (because the latter is a k-retractible extension of $\widetilde{G}$). Thus all these groups are p-groups of finite exponent. As before, we see that N_k/N_{k+1} embeds in the product of p-groups of bounded exponents, hence the assertion. The fact that the quotient $\widetilde{G}/N_{k+1}$ is then also a p-group of bounded exponent follows directly. This proves statement 3.

The proof of statement 4 is similar to that of statement 3. $\qquad\square$

3.4. **The construction and retra-products.** Given any n, we construct an n-retractible extended simplex of groups $\mathbf{G}^+$, over the simplex Δ of arbitrary dimension, as follows:

   1. We put the trivial group on the simplex Δ.

2. We put an arbitrary group $\mathbf{G}(s)$ on each codimension 1 face s of Δ.

3. Suppose we have already defined groups $\mathbf{G}(\eta)$ (and inclusion maps between them) for faces of codimension strictly less than k. Then for a face τ of codimension $k \leq \dim \Delta$, the group $\mathbf{G}(\tau)$ is the minimal n-retractible extension, as in Theorem 20, of the simplex of groups over the link simplex Δ_τ made of the already defined groups $\mathbf{G}(\eta)$, via the canonical correspondence between the faces of Δ_τ and the faces $\eta \subset \Delta$ containing τ.

4. Finally, we take as $\mathbf{G}(\emptyset)$ the minimal n-retractible extension of the so far obtained simplex of groups $\mathbf{G}$ over Δ.

Definition 3.25 (Retra-product). We will call the group $\mathbf{G}(\emptyset)$ of any simplex of groups $\mathbf{G}^+$ obtained as in the construction above the n-*retra-product* of the (finite) family of groups $\mathbf{G}(s) : s \in \mathcal{S}_\Delta$. Note that this operation makes sense for any finite family of groups.

Clearly, the groups $\mathbf{G}(\tau)$ obtained in the construction above are all the n-retra-products of the corresponding families of groups $\mathbf{G}(s) : s \in \mathcal{S}_\Delta, s \supset \tau$.

Rephrasing Theorem 3.23 in the context of retra-products we get the following properties of this operation, for an arbitrary natural number n:

   (1) the n-retra-product of finite groups is finite;
   (2) the n-retra-product of p-groups of bounded exponent is a p-group of bounded exponent.

It follows that the n-retra-product of finite p-groups is a finite p-group. In particular, we get the following.

Corollary 3.26. *Let* $\mathbf{G}$ *be a non-extended simplex of groups obtained as in the construction above, out of groups* $\mathbf{G}(s)$ *being finite p-groups. Then all groups* $\mathbf{G}(\tau)$ *in this simplex are finite p-groups.*

Remark 3.27. The construction of this subsection was first used in [9], in the 2-retractible case. For the purposes of this paper, it suffices to consider the case when each codimension one face of the simplex is assigned the cyclic group $\mathbb{Z}_p$ of order p. Even though the construction of the n-retra-products is in principle explicit, one rapidly loses track of the groups arising.

The n-retra-product of two groups $\mathbb{Z}_2$, occurring at faces of codimension 2, is the dihedral group D_{2^n} of order 2^{n+1}. We do not even know the orders of the n-retra-products of three copies of $\mathbb{Z}_2$, except the case $n = 2$ when this order is 2^{14}.

Note that if $\mathbf{G}$ is a non-extended simplex of groups corresponding to $\mathbf{G}^+$, obtained by the construction above, then its direct limit extension $\mathbf{G}^+_{dir}$ is different from $\mathbf{G}^+$, and in particular the direct limit $\mathbf{G}^+_{dir}(\emptyset)$ is different from the n-retra-product $\mathbf{G}(\emptyset)$. This motivates the following.

Definition 3.28 (Free retra-product). The direct limit of a non-extended simplex of groups $\mathbf{G}$ obtained by the construction above will be called the *free n-retra-product* of the (finite) family of the codimension 1 groups $\mathbf{G}(s) : s \in \mathcal{S}_\Delta$.

In the next subsection we will deal with free n-retra-products of finite groups, showing that for $n \geq 2$ they are infinite, and for $n \geq 3$ they are non-elementary word-hyperbolic.

3.5. Simplicial non-positive curvature, word-hyperbolicity, and the proof of Theorem 3.1.

To show that there are non-elementary word-hyperbolic groups arising from the construction of the previous subsection, we will use results of [10] concerning simplicial non-positive curvature.

Recall from [10] that the systole $sys(K)$ of a simplicial complex K is the smallest number of 1-simplices in any full subcomplex of K homeomorphic to the circle. A simplicial complex K is k-large if its systole and the systoles of links of all simplices in K are all at least k. Simplicial complexes whose all links are k-large, for some fixed k, are the analogs of metric spaces with curvature bounded above. If they are additionally simply connected, we call them *k-systolic complexes*. All results of this subsection are corollaries to the following.

Proposition 3.29. *Suppose $\mathbf{G}^+$ is an n-retractible extended simplex of groups. Then the development $d\mathbf{G}^+$ is $2(n + 1)$-large.*

We skip the proof of the proposition until the end of the subsection, first deriving (and making comments on) its consequences. In particular, we show how this proposition, together with the results of the previous subsection, implies Theorem 3.1.

Corollary 3.30. *Suppose $\mathbf{G}$ is a non-extended simplex of groups whose links $\mathbf{G}_\sigma^+$ are n-retractible, and let $\mathbf{G}_{dir}^+$ be the direct limit extension of $\mathbf{G}$.*

  (1) *If $n \geq 2$ then the development $d\mathbf{G}_{dir}^+$ is contractible.*
  (2) *If $n \geq 2$ and the codimension 1 groups $\mathbf{G}(s)$ are non-trivial, then $d\mathbf{G}_{dir}^+$ and the direct limit $\mathbf{G}_{dir}(\emptyset)$ are both infinite.*
  (3) *If $n \geq 3$ then the 1-skeleton of the development $d\mathbf{G}_{dir}^+$, equipped with the polygonal metric, is Gromov-hyperbolic.*
  (4) *If $n \geq 3$ and all the groups $\mathbf{G}(\sigma)$ are finite and nontrivial, then the group $\mathbf{G}_{dir}(\emptyset)$ is non-elementary word-hyperbolic, except when the underlying simplex is the 1-simplex and the vertex groups are of order 2 (in which case it is the infinite dihedral group).*
  (5) *If $n \geq 2$ and all the groups $\mathbf{G}(\sigma)$ are finite, then the group $\mathbf{G}_{dir}(\emptyset)$ is virtually torsion-free.*

Proof. First, note that links in $d\mathbf{G}_{dir}^+$ are isomorphic to the developments of the n-retractible link simplices of groups $\mathbf{G}_\sigma$ (see Lemma 9(1)), and thus are $2(n+1)$-large. Since $d\mathbf{G}_{dir}^+$ is the universal cover of $\mathbf{G}$, it is simply connected and hence $2(n+1)$-systolic. For $n \geq 2$, this means that $d\mathbf{G}_{dir}^+$ is 6-systolic, and thus it is contractible by [10, Theorem 4.1(1)]. This proves (1).

To get assertion (2), note that it follows from [10, Proposition 18(2)], that the group $\mathbf{G}_{dir}(\emptyset)$ has a nontrivial subgroup that acts on the development $d\mathbf{G}_{dir}^+$ freely. Thus this development is a classifying space for this subgroup and, since it has finite dimension,

the subgroup has to be infinite. See also [17, Corollary 4.3], for a more elementary argument.

For parts (3) and (4), note first that $n \geq 3$ implies 8-systolicity of the development $d\mathbf{G}_{dir}^+$. Part (3) follows then from [10, Theorem 2.1].

Finally, under the assumptions of (4), the group $\mathbf{G}_{dir}(\emptyset)$ acts on $d\mathbf{G}_{dir}^+$ properly discontinuously and cocompactly. Thus it follows from (3) that $\mathbf{G}_{dir}(\emptyset)$ is word-hyperbolic. If the underlying simplex of $|\mathbf{G}|$ is 1-dimensional, the group acts geometrically on the tree $d\mathbf{G}_{dir}^+$, and hence is virtually free non-abelian (except the mentioned case). If the dimension of the underlying simplex $|\mathbf{G}|$ is greater than 1, the fact that the group is non-elementary follows from [18, Theorem 5.6 and Remark 2 at the end], where it is shown that this group has one end.

The non-elementariness above can be also shown directly, by noting that the groups as in (4) contain as a subgroup the free product of three nontrivial finite groups (namely codimension 1 groups in any vertex unfolding of certain three pairwise disjoint sides). We do not include the details of this argument.

In order to prove (5), recall that by Theorem 3.23 there exists a finite extension $\mathbf{G}_{min}(\emptyset)$ of $\mathbf{G}$. Since the canonical homomorphism $\lambda : \mathbf{G}_{dir}(\emptyset) \to \mathbf{G}_{min}(\emptyset)$ is injective on the local groups, the kernel H of λ acts freely on the development $d\mathbf{G}_{dir}^+$ and the index $|\mathbf{G}_{dir}(\emptyset) : H|$ is finite. Suppose that H contains a non-trivial element h of finite order p; without loss of generality, we can assume that p is prime. The space $d\mathbf{G}_{dir}^+$ is finite-dimensional by definition, and contractible by part (1), therefore, by Smith fixed point theory (see [3, Thm. III.7.11]), the induced action of $\langle h \rangle$ on this space has a fixed point. This yields a contradiction with the freeness of the action of H on $d\mathbf{G}_{dir}^+$. $\qquad \square$

Restricting to free n-retra-products, we immediately get the following.

Corollary 3.31. *If $n \geq 2$ then the free n-retra-product of (at least two) nontrivial groups is infinite and virtually torsion-free. If $n \geq 3$ then the free n-retra-product of (at least two) nontrivial finite groups is non-elementary word-hyperbolic, except the product of two groups of order 2.*

Remark 3.32. Note that the construction in the present paper gives new families of k-systolic groups, for arbitrary k, different from those constructed in [10, Sections 17–20]. These are the free n-retra-products of arbitrary finite groups, where $2(n+1) \geq k$.

Proof of Theorem 3.1. Let $\mathbf{G}$ be the 3-retractible $(n+1)$-simplex of groups obtained by the construction of Subsection 3.4, with all codimension 1 groups $\mathbf{G}(s)$ isomorphic to the cyclic group $\mathbb{Z}_p$ of order p. Let G be the direct limit of $\mathbf{G}$, i.e. the free 3-retra-product of $n+2$ copies of $\mathbb{Z}_p$. By Corollary 3.30, G is then non-elementary word hyperbolic.

Choose a generator for each codimension 1 subgroup $\mathbf{G}(s)$ of G (i.e. for each factor of the above free 3-retra-product). Let S be the set formed of these generators. Then S consists of $n+2$ elements, and it generates G since the union $\cup\{\mathbf{G}(s) : s \in |\mathbf{G}|\}$ generates G. For any proper subset $T \subset S$, the subgroup $\langle T \rangle < G$ generated by T

coincides with one of the local groups of $\mathbf{G}$. More precisely, for $\tau = \cap\{s : s \in T\}$ we have $\langle T \rangle = \mathbf{G}(\tau)$. By Corollary 3.25, $\langle T \rangle$ is then a finite p-group, which completes the proof. $\qquad\square$

It only remains now to prove Proposition 3.29. We will use the following lemma in the proof.

Lemma 3.33. *Let γ be a loop in the 1-skeleton of a simplicial complex K which has the minimum length L amongst all loops in the 1-skeleton of K in the same free homotopy class. Then any lift $\widetilde{\gamma}$ of γ to the universal cover $\widetilde{K}$ of K has the property that it minimizes distance measured in the 1-skeleton of $\widetilde{K}$ between any two points whose distance in $\widetilde{\gamma}$ is at most L.*

Proof. Let t be a deck transformation of $\widetilde{K}$ that acts as a translation by the distance L on the subcomplex $\widetilde{\gamma}$. Suppose that $\widetilde{\gamma}$ does not have the property claimed. Then there exist vertices p and q of $\widetilde{\gamma}$ such that q lies on the segment from p to tp and such that the distance between p and q in the 1-skeleton of $\widetilde{K}$ is strictly smaller than the distance between them in $\widetilde{\gamma}$. Let α be the segment of $\widetilde{\gamma}$ between p and q, and let α' be a path of shorter length in the 1-skeleton of $\widetilde{K}$ from p to q. Denote also by β be the segment of $\widetilde{\gamma}$ between q and tp. Since $\widetilde{K}$ is simply connected, α and α' are homotopic relative to their endpoints. This homotopy, after projecting to K, yields a homotopy between γ and the loop obtained by projecting $\alpha' \cup \beta$. Since the latter loop is strictly shorter, we get a contradiction. $\qquad\square$

Proof of Proposition 3.29. The proof is by induction on d, the dimension of the simplex $|\mathbf{G}|$, followed by induction on n. For $d = 0$ there is nothing to prove. Let $\mathbf{G}^+$ be an n-retractible d-dimensional simplex of groups. Then links $\mathbf{G}^+_\sigma$ are also n-retractible (see Remark 17(3)), and thus by induction on d their developments $d\mathbf{G}^+_\sigma$ are $2(n+1)$-large. Thus the same holds for links of $d\mathbf{G}^+$.

By [10, Corollary 1.5], a simplicial complex X with k-large links is k-large iff the length of the shortest loop in the 1-skeleton of X which is homotopically nontrivial in X is at least $2(n+1)$. We thus need to show this for $X = d\mathbf{G}^+$.

Let γ be a homotopically nontrivial polygonal loop in $d\mathbf{G}^+$ of the shortest length. Now we start the induction on n. It has been shown in [9, Proposition 4.3(3)], that the development of any 1-retractible extended simplex of groups is a flag complex. Hence the length of γ is at least 4. This completes the case $n = 1$ for all d. Clearly an n-retractible complex is $(n-1)$-retractible, and so by induction on n, the length of γ is at least $2n$.

Let $\widetilde{X}$ be the universal cover of $X = d\mathbf{G}^+$, let $\widetilde{\gamma}$ be a lift of γ to $\widetilde{X}$, and let $v_0, \ldots, v_{n+1}$ be some consecutive vertices on $\widetilde{\gamma}$. By Lemma 3.33, we see that $\widetilde{\gamma}$ minimizes distances between the vertices $v_i : 0 \leq i \leq n+1$ in the 1-skeleton of $\widetilde{X}$. In particular the distance between v_0 and v_{n+1} is equal to $n+1$. Let δ denote the segment of $\widetilde{\gamma}$ between v_0 and v_{n+1}, and let δ' be the segment of $\widetilde{\gamma}$ that starts at v_{n+1} and projects to the segment of γ complementary to the projection of δ.

By symmetry of $d\mathbf{G}^+$, we may assume that v_1 is a vertex of the simplex $[|\mathbf{G}|, 1] \subset d\mathbf{G}^+$. Consider the fundamental domain $D \subseteq \widetilde{X}$ for the group $\widetilde{G}_{v_1,\ldots,v_n} = d_{v_1,\ldots,v_n}(\emptyset)$ obtained recursively as $d_{v_n}\ldots d_{v_1}|\mathbf{G}|$ in the way described just before Lemma 7. By Lemma 7, D is a strict fundamental domain. It contains the vertices $v_1,\ldots,v_n$ in its interior (i.e., outside the boundary ∂D), and thus contains also v_0 and v_{n+1}. We identify D with the quotient $\widetilde{G}_{v_1,\ldots,v_n}\backslash X$. We then look at the projection of $\delta \cup \delta'$ to D. Since the distance in $\widetilde{X}$ between v_0 and v_{n+1} is $n+1$, their distance in D is also $n+1$ (the distance in the quotient cannot increase, while that in the subcomplex cannot increase). It follows that the length of the projection of δ' to D is at least $n+1$, and so the length of γ is at least $2(n+1)$. $\qquad\square$

3.6. **The fixed point property.** We now pass to proving Theorem 3.2. The proof will use the following lemma. The proof of a related result, for contractible CW-complexes, can be found in [13]. We remind the reader that "mod-p acyclic" means "having the same mod-p Čech cohomology as a point".

Lemma 3.34. *Fix an integer $n > 0$, let $Y_1,\ldots,Y_n$ be closed subspaces of a space X, let $Y = \bigcup_{i=1}^n Y_i$ and let $A = \bigcap_{i=1}^n Y_i$, the union and intersection of the Y_i respectively. Suppose that for all subsets $I \subset \{1,\ldots,n\}$ with $1 \le |I| < n$, the intersection $\bigcap_{i\in I} Y_i$ is mod-p acyclic. Then the reduced mod-p Čech cohomologies of Y and A are isomorphic, with a shift in degree of $n-1$. More precisely, for each m there is an isomorphism $\widetilde{H}^m(Y) \cong \widetilde{H}^{m-n+1}(A)$.*

Proof. In the case when $n = 1$, we have that $Y = A$ and the assertion is trivially true. Now suppose that $n \ge 2$. For $1 \le i \le n-1$, let $Z_i = Y_i \cap Y_n$, let $Z = \bigcup_i Z_i$, and let $Y' = \bigcup_{i=1}^{n-1} Y_i$. By definition, $Y - Y' = Y_n - Z$, and so by the strong form of excision that holds for Čech cohomology (see the end of section 3.3 of [7]), it follows that $H^*(Y, Y') \cong H^*(Y_n, Z)$.

By induction on n, we see that for each m, $\widetilde{H}^m(Z) \cong \widetilde{H}^{m-n+2}(A)$. Also by induction, we see that Y' is mod-p acyclic, since $\bigcap_{i=1}^{n-1} Y_i$ is mod-p acyclic by hypothesis. Since $n \ge 2$, the hypotheses also imply that Y_n is mod-p acyclic. Hence the long exact sequence in reduced cohomology for the pair (Y_n, Z) collapses to isomorphisms, for all i, $\widetilde{H}^{i-1}(Z) \cong H^i(Y_n, Z)$. Similarly, the long exact sequence in reduced cohomology for the pair (Y, Y') collapses to isomorphisms, for all i, $H^i(Y, Y') \cong \widetilde{H}^i(Y)$. Putting these isomorphisms together gives an isomorphism, for all i, of reduced cohomology groups $\widetilde{H}^{i-1}(Z) \cong \widetilde{H}^i(Y)$. The claimed result follows. $\qquad\square$

Proof of Theorem 3.2. Suppose that X is a p-acyclic G-space of finite covering dimension. Let $g_1,\ldots,g_{n+2}$ be the elements of S, and let Y_i be the points of X that are fixed by g_i. By Smith theory, the fixed point set of the action of any finite p-group on X is mod-p acyclic. For an explicit reference, see Theorem III.7.11 of Bredon's book [3], noting that "finite covering dimension" implies Bredon's hypothesis "finitistic" (see

p. 133 of [3]), and that "X is mod-p acyclic" is equivalent to Bredon's hypothesis "the pair $(X, \emptyset)$ is a mod-p Čech cohomology 0-disk". Hence the subspaces Y_i satisfy the hypotheses of Lemma 3.34. The global fixed point set A for the action of G on X is equal to the intersection $A = \bigcap_{i=1}^{n+2} Y_i$. If A is not mod-p acyclic, then for some $m \geq -1$, the reduced cohomology group $\widetilde{H}^m(A)$ is non-zero. By Lemma 3.34, it follows that the union $Y = \bigcup_{i=1}^{n+2} Y_i$ has a non-vanishing reduced cohomology group $\widetilde{H}^j(Y)$ for some $j \geq n$. Hence the relative cohomology group $H^{j+1}(X, Y)$ is non-zero for some $j \geq n$, and so X must have covering dimension at least $n + 1$. □

4. Finitely presented groups that fail to act

In this section we establish the validity of Template $\mathbf{NA}_{fp}$. We also show the triviality of actions by diffeomorphisms of certain groups, see Lemma 4.5. This immediately implies Proposition 1.4 and provides the input to Template $\mathbf{NA}_{fp}$ that is needed to prove Theorem 1.5. Our proof of Lemma 4.5 was inspired by recent work of Bridson and Vogtmann [5].

Definition 4.1. A sequence of groups and monomorphisms, $(G_n; \xi_{n,j})$ $(n \in \mathbb{N}, j = 1, \ldots, J)$, is called *a recursive system* if

  (i) each G_n has a presentation $\langle \mathcal{A}_n \parallel R_n \rangle$ with $\mathcal{A}_n$ finite and $\bigcup_n R_n \subset \mathcal{A}^*$ recursively enumerable, where $\mathcal{A} = \sqcup_n \mathcal{A}_n$, and
  (ii) each monomorphism $\xi_{n,j} : G_n \to G_{n+1}$ is defined by a set of words $S_{n,j} = \{w_{n,j,a} \in \mathcal{A}_{n+1}^* \mid a \in \mathcal{A}_n\}$ such that $w_{n,j,a} = \xi_{n,j}(a)$ in G_{n+1}, with $\bigcup_{n,j} S_{n,j} \subset \mathcal{A}^*$ recursively enumerable.

We shall be interested only in sequences where, for each sufficiently large integer n, G_{n+1} is generated by the union of the images of the $\xi_{n,j}$. And in our applications we shall need only the case $J = 2$.

Examples 4.2. The following are recursive systems.
  (1) Define $G_n = \mathrm{SL}(n, \mathbb{Z})$, set $J = 2$, and define $\xi_{n,1}$ and $\xi_{n,2}$ to be the embeddings $\mathrm{SL}(n, \mathbb{Z}) \to \mathrm{SL}(n+1, \mathbb{Z})$ defined by

$$\xi_{n,1}(M) = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \xi_{n,2}(M) = \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \quad \text{for all } M \in \mathrm{SL}(n, \mathbb{Z}).$$

  (2) Writing $Alt(n)$ to denote the alternating group consisting of even permutations of $\mathbf{n} = \{1, \ldots, n\}$ and taking $J = 2$, define $G_n = Alt(n)$ and define $\xi_{n,1}, \xi_{n,2} : Alt(n) \to Alt(n+1)$ to be the embeddings induced by the maps $I_{n,1}, I_{n,2} : \mathbf{n} \to \mathbf{n+1}$ defined by $I_{n,1} : k \to k$ and $I_{n,2} : k \to k + 1$.

Lemma 4.3. *Let $(G_n; \xi_{n,j})$ $(n \in \mathbb{N}, j = 1, \ldots, J)$ be a recursive system of non-trivial groups and monomorphisms $\xi_{n,j} : G_n \to G_{n+1}$, and suppose that there exists n_0 so that for each $n \geq n_0$, G_{n+1} is generated by $\bigcup_j \xi_{n,j}(G_n)$. Then there exists a finitely presented*

group G_ω which for each $n \geq n_0$, contains two isomorphic copies of G_n so that G_ω is the normal closure of the union of these two subgroups.

Proof. By renumbering, we may assume that $n_0 = 1$. Consider the free product

(2) $$G = \overset{\infty}{\underset{n=1}{*}} G_n,$$

and for each $j \in J$ let $\theta_j : G \to G$ be the injective endomorphism of G whose restriction to G_n is $\xi_{n,j}$.

Now, let H be the multiple HNN-extension of G corresponding to these endomorphisms:

$$H = \langle G, t_1, \ldots, t_j \parallel t_j g t_j^{-1} = \theta_j(g), \ g \in G, j \in J \rangle .$$

In the notation of Definition 4.1, this has presentation

$$\langle \mathcal{A}, t_1, \ldots, t_j \parallel R_n \ (n \in \mathbb{N}), \ t_j a t_j^{-1} w_{n,j,a}^{-1} \ (n \in \mathbb{N}, j \in J, a \in \mathcal{A}_n) \rangle.$$

By hypothesis, this is a recursive presentation. Moreover, since the images of the $\xi_{n,j}$ generate G_{n+1}, the group H is generated by the finite set $\mathcal{A}_1 \cup \{t_1, \ldots, t_J\}$.

Now, by the Higman embedding theorem (see [14, IV.7]), H can be isomorphically embedded into a finitely presented group B. Suppose that B is generated by elements $x_1, \ldots, x_l$. Without loss of generality we can and do assume that each of $x_1, \ldots, x_l$ has infinite order. Indeed, to ensure this one can if necessary replace B by the free product $B * \mathbb{Z}$ of B with the infinite cyclic group generated by z, which is generated by the elements $z, zx_1, \ldots, zx_l$ of infinite order. Choose an element of infinite order $y \in G \leq B$ and a subgroup $F \leq G$ such that F is free of rank l and

(3) $$F \cap \langle y \rangle = \{1\}, \ F \cap \langle x_i \rangle = \{1\} \ \text{for } i = 1, \ldots, l.$$

Such a choice is possible because G is a free product of infinitely many non-trivial groups and a cyclic subgroup of B can intersect at most one free factor non-trivially.

Consider, now, the iterated HNN-extension of B:

$$L = \langle B, s_1, \ldots, s_l \parallel s_i x_i s_i^{-1} = y, \ i = 1, \ldots, l \rangle.$$

Let $\{f_1, \ldots, f_l\}$ be free basis of F. By (3) and Britton's lemma ([14, IV.2]), the subgroup of L generated by $s_1, \ldots, s_l$ and F is freely generated by the elements $s_1, \ldots, s_l$, $f_1, \ldots, f_l$. Let L' be a copy of L and let $s_1', \ldots, s_l', f_1', \ldots, f_l'$ denote the copies of the corresponding elements. Finally we obtain the group that we seek by defining

$$G_\omega = \langle L, L' \parallel s_i = f_i', f_i = s_i', \ i = 1, \ldots, l \rangle.$$

The group G_ω is infinite and finitely presented by construction.

A key feature in our construction is that for all $k \leq n$ the free factor G_k of G is conjugate in H (and hence in G_ω) to a subgroup of G_n, and for $k > n$ the conjugates of G_n by positive words of length $k - n$ in the letters t_j generate G_k. Thus G is the normal closure of each G_n. Likewise G' is the normal closure of G_n'. All that remains is to observe that G_ω is generated by the set $\{x_1, \ldots, x_l, s_1, \ldots, s_l, x_1', \ldots, x_l', s_1', \ldots, s_l'\}$,

each of whose elements is conjugate to an element of G or G'. Thus G_ω is the normal closure of $G_n \cup G'_m$ for every $n, m \geq 1$. $\qquad\square$

The following theorem establishes the validity of Template $\mathbf{NA}_{fp}$.

Theorem 4.4. *If the groups G_n satisfy the conditions of Template $\mathbf{NA}_{fp}$, then the finitely presented group G_ω constructed in Lemma 4.3 cannot act non-trivially on any $X \in \mathcal{X}$.*

Proof. Suppose that G_ω acts on a space $X \in \mathcal{X}$. Then $X \in \mathcal{X}_m$ for some $m \in \mathbb{N}$ and we have a homomorphism $\alpha : G_\omega \to \mathrm{Homeo}(X)$ that we want to prove is trivial. By hypothesis, there is some G_n that cannot act non-trivially on X. Hence, in the notation of the preceding lemma, $\alpha(G_n) = \alpha(G'_n) = \{1\}$. Therefore the kernel of α is the whole of G_ω. $\qquad\square$

Lemma 4.5. *If G is a simple group that contains a copy of $\mathbb{Z}_p^{n+1}$, then G cannot act effectively by diffeomorphisms on any smooth p-acyclic manifold X of dimension at most n.*

Proof. Let E be the subgroup of G isomorphic to $(\mathbb{Z}_p)^{n+1}$. Fix a Riemannian metric on X that is compatible with the smooth structure and, given an action of G, average the metric over the action of E to ensure that the action of E is by isometries.

Smith theory tells us that the action of E on X has mod-p acyclic fixed point set (Theorem III.7.11 of [3]), and so there is a point x fixed by E. Taking derivatives at x, we obtain an action of E on the tangent space $T_x(X)$. But E has no faithful linear representations of dimension less than or equal to n, and so some non-identity element $h \in E$ acts trivially on $T_x(X)$. Since the action of E is by isometries, the exponential map is E-equivariant and so h acts trivially on an open ball about x. But the fixed point set for any isometry of X is a closed submanifold, and so it follows that h fixes the whole of X.

Thus h lies in the kernel of the map $G \to \mathrm{Diffeo}(X)$. Since G is simple, it follows that G acts trivially. $\qquad\square$

Proof of Theorem 1.5 The preceding lemma tells us that, given any prime p and positive integer n, any alternating group $Alt(m)$ with m sufficiently large cannot act non-trivially by diffeomorphisms on a smooth p-acyclic manifold of dimension less than n. It follows that the group G_ω obtained by applying Theorem 4.4 to the recursive system in Example 4.2(2) cannot act non-trivially by diffeomorphisms on a p-acyclic manifold of any dimension, for all primes p.

Remark 4.6. (1) For each fixed prime p, C. Röver constructed a finitely presented simple group containing, for each n, a copy of $(\mathbb{Z}_p)^n$ [22]. By Lemma 4.5, such a group cannot act non-trivially by diffeomorphisms on any mod-p acyclic manifold.

Certain of the finitely presented simple groups introduced by Thompson [23] and Higman [8] contain a copy of every finite group, so these cannot act by diffeomorphisms on a mod-p acyclic manifold for any p.

(2) It should be clear from the architecture of our proof how suitable variations on Lemma 4.5 will give rise to analogues of Proposition 1.4 and Theorem 1.5. The work of Bridson and Vogtmann [5] provides several such variations.

References

[1] G. Arzhantseva, A. Minasyan, D. Osin, *The SQ-universality and residual properties of relatively hyperbolic groups*, J. of Algebra **315** (2007), 165–177.

[2] A.K. Barnhill, *The* FA_n *conjecture for Coxeter groups*, Algebr. Geom. Topol. **6** (2006), 2117–2150.

[3] G.E. Bredon, *Introduction to compact transformation groups*, Academic Press, 1972.

[4] M. Bridson, A. Haefliger, *Metric spaces of nonpositive curvature*, Grundlehren der Math. Wiss. **319**, Springer, 1999.

[5] M. Bridson, K. Vogtmann, *Actions of automorphism groups of free groups on spheres and generalized manifolds*, preprint, 2008. ArXiv:0803.2062

[6] K.S. Brown, R. Geoghegan, *An infinite-dimensional torsion-free* FP_∞ *group*, Invent. Math. **77** (1984), no. 2, 367–381.

[7] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[8] G. Higman, *Finitely Presented Infinite Simple Groups*, Notes on Pure Mathematics **8**, ANU, Canberra, 1974.

[9] T. Januszkiewicz, J. Świątkowski, *Hyperbolic Coxeter groups of large dimension*, Comment Math. Helv. **78** (2003), 555–583.

[10] T. Januszkiewicz, J. Świątkowski, *Simplicial nonpositive curvature*, Publ. Math. IHES **104** (2006), 1–85.

[11] T. Januszkiewicz, J. Świątkowski, *Filling invariants for systolic complexes and groups*, Geometry and Topology **11** (2007), 727–758.

[12] P.H. Kropholler, *On groups of type* $(\mathrm{FP})_\infty$, J. Pure Appl. Algebra **90** (1993), no. 1, 55–67.

[13] I.J. Leary, *On finite subgroups of groups of type VF*, Geometry and Topology **9** (2005), 1953–1976.

[14] R. Lyndon, P. Schupp, *Combinatorial Group Theory*, Springer-Verlag, 1977.

[15] A. Minasyan, *On residualizing homomorphisms preserving quasiconvexity*, Comm. in Algebra **33** (2005), no. 7, 2423–2463.

[16] A.Yu. Ol'shanskii, *On residualing homomorphisms and G-subgroups of hyperbolic groups*, Intern. J. of Algebra and Comput. **3** (1993), no. 4, 365–409.

[17] D. Osajda, *Connectedness at infinity of systolic complexes and groups*, Groups, Geometry and Dynamics **1** (2007), 183–203.

[18] D. Osajda, *Ideal boundary of 7-systolic complexes and groups*, Algebr. Geom. Topol. **8** (2008), no. 1, 81–99.

[19] D.V. Osin, *Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems*, Mem. Amer. Math. Soc. **179** (2006), no. 843, vi+100 pp.

[20] D.V. Osin, *Weakly amenable groups*, Computational and statistical group theory (Las Vegas, NV/Hoboken, NJ, 2001), 105–113, Contemp. Math. **298**, Amer. Math. Soc., Providence, RI, 2002.

[21] A. Robert, *Introduction to the representation theory of compact and locally compact groups*, London Mathematical Society Lecture Note Series **80** (1983), Cambridge University Press, Cambridge-New York, ix+205 pp.

[22] C. Röver, *Constructing finitely presented simple groups that contain Grigorchuk groups*, J. Alg. **220** (1999), 284–313.

[23] R.J. Thompson, *Embeddings into finitely generated simple groups which preserve the word problem*, in "Word Problems II" (S.I. Adian, W.W. Boone and G. Higman, eds.), North-Holland, Amsterdam, 1980.

(G. Arzhantseva) UNIVERSITÉ DE GENÈVE, SECTION DE MATHÉMATIQUES, 2-4 RUE DU LIÈVRE, CASE POSTALE 64, 1211 GENÈVE 4, SWITZERLAND
E-mail address: `Goulnara.Arjantseva@math.unige.ch`

(M.R. Bridson) MATHEMATICAL INSTITUTE, 24-29 ST GILES', OXFORD, UK
E-mail address: `bridson@maths.ox.ac.uk`

(T. Januszkiewicz) DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, 231 W. 18TH AVE., COLUMBUS, OH 43210, USA AND THE MATHEMATICAL INSTITUTE OF POLISH ACADEMY OF SCIENCES. ON LEAVE FROM INSTYTUT MATEMATYCZNY, UNIWERSYTET WROCŁAWSKI
E-mail address: `tjan@math.ohio-state.edu`

(I. J. Leary) DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, 231 W. 18TH AVE., COLUMBUS, OH 43210, USA
E-mail address: `leary@math.ohio-state.edu`

(A. Minasyan) UNIVERSITÉ DE GENÈVE, SECTION DE MATHÉMATIQUES, 2-4 RUE DU LIÈVRE, CASE POSTALE 64, 1211 GENÈVE 4, SWITZERLAND. ON LEAVE FROM SCHOOL OF MATHEMATICS, UNIVERSITY OF SOUTHAMPTON, HIGHFIELD, SOUTHAMPTON, SO17 1BJ, UNITED KINGDOM.
E-mail address: `aminasyan@gmail.com`

(J. Świątkowski) INSTYTUT MATEMATYCZNY, UNIWERSYTET WROCLAWSKI, PL. GRUNWALDZKI 2/4, 50-384 WROCLAW, POLAND
E-mail address: `swiatkow@math.uni.wroc.pl`