# Boundaries of systolic groups

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**Abstract**

For all systolic groups we construct boundaries which are $EZ$–structures. This implies the Novikov conjecture for torsion–free systolic groups. The boundary is constructed via a system of distinguished geodesics in a systolic complex, which we prove to have coarsely similar properties to geodesics in $CAT(0)$ spaces.

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# Contents



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# 1 Introduction

There are many notions of boundaries of groups used for various purposes. In this paper we focus on the notions of $Z$–structure and $EZ$–structure introduced by Bestvina [2] and studied e.g. in [9], [13]. Our main result is the following.

**Theorem A** (Theorem 6.3)**.** *Let a group $G$ act geometrically by simplicial automorphisms on a systolic complex $X$. Then there exists a compactification $\overline{X} = X \cup \partial X$ of $X$ satisfying the following:*

1. *$\overline{X}$ is a Euclidean retract (ER),*
2. *$\partial X$ is a $Z$–set in $\overline{X}$,*
3. *for every compact set $K \subset X$, $(gK)_{g\in G}$ is a null sequence,*
4. *the action of $G$ on $X$ extends to an action, by homeomorphisms, of $G$ on $\overline{X}$.*

A group $G$ as in Theorem A is called a *systolic group.* It is a group acting *geometrically* (i.e. cocompactly and properly discontinuously) by simplicial automorphisms on a *systolic complex*—contractible simplicial complex satisfying some local combinatorial conditions. Systolic complexes were introduced by Januszkiewicz–Świątkowski [16] and, independently, by Haglund [14] and by Chepoi [6] (in Section 2 we give some background on them). Systolic complexes (groups) have many properties of non–positively curved spaces (groups). There are systolic complexes that are not $CAT(0)$ when equipped with the path metric in which every simplex is isometric to the standard Euclidean simplex. On the other hand, there are systolic groups that are not hyperbolic, e.g. $\mathbb{Z}^2$. Summarizing, systolic setting does not reduce to the $CAT(0)$ or to the hyperbolic one — it turns out that systolic groups form a large family: allow various combinatorial constructions [14],[16],[1] and provide the discipline with new range of examples frequently with unexpected properties [17],[1]. We also believe that eventually both systolic complexes and $CAT(0)$ cubical ones will be placed among a wider family of combinatorially non–positively curved contractible cell complexes.

Here we give the other definitions that appear in the statement of Theorem A. A compact space is a *Euclidean retract* (or ER) if it can be embedded in some Euclidean space as its retract. A closed subset $Z$ of a Euclidean retract $Y$ is called a *$Z$–set* if for every open set $U \subset Y$, the inclusion $U \setminus Z \hookrightarrow U$ is a homotopy equivalence. A sequence $(K_i)_{i=1}^{\infty}$ of subsets of a topological space $Y$ is called a *null sequence* if for every open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $Y$ all but finitely many $K_i$ are *$\mathcal{U}$–small*, i.e. for all but finitely many $j$ there exist $i(j)$ such that $K_j \subset U_{i(j)}$.

Conditions 1, 2 and 3 of Theorem A mean (following [2], where only free actions are considered, and [9]) that any systolic group $G$ admits a *$Z$–structure* $(\overline{X}, \partial X)$. The notion of an *$EZ$–structure*, i.e. a $Z$–structure with additional property 4 was explored by Farrell–Lafont [13] (in the case of a free action).

Bestvina [2] showed that some local homological invariants of the boundary $\partial X$ are related to cohomological invariants of the group. In particular, the dimension of the boundary is an invariant of the group i.e. it does not depend on the $Z$–structure we choose. This was generalized by Dranishnikov [9] to the case of geometric actions. It should be emphasized that the homeomorphism type of the boundary is not a group invariant (but the shape is an invariant [2]), as the Croke–Kleiner examples of visual boundaries of some $CAT(0)$ spaces exhibit [7]. Carlsson–Pedersen [5] and Farrell–Lafont [13] proved that existence of an $EZ$–structure on a torsion–free group $G$ implies that the Novikov conjecture is true for $G$. Thus, by Theorem A, we get the following.

**Corollary.** *Torsion–free systolic groups satisfy the Novikov conjecture.*

There are only few classes of groups for which a $Z$–structure $(\overline{X}, \partial X)$ has been found (and even fewer for which an $EZ$–structure is known). The most important examples are: hyperbolic groups [3] —with $X$ being the Rips complex and $\partial X$ being the Gromov boundary of $G$; $CAT(0)$ groups —with $X$ a $CAT(0)$ space and $\partial X$ the visual boundary of $X$; relatively hyperbolic groups whose parabolic subgroups admit a $Z$–structure [8]. Bestvina [2] asked whether every group $G$ with finite $K(G, 1)$ has a $Z$–structure.

The question whether for every systolic group there exists an $EZ$–structure was posed by Januszkiewicz and Świątkowski in 2004. Theorem A answers affirmatively this question.

We hope that, similarly to the hyperbolic and $CAT(0)$ cases, our boundaries will be also useful for purposes other than the ones mentioned above. In particular we think that splittings of systolic groups can be recognized through the topology of the boundary, as in e.g. [4], [18]. Studying more refined structures on the boundary could help in obtaining rigidity results for some systolic groups.

The essential point of our construction is the choice of the system of *good geodesics* (derived from the system of *Euclidean geodesics*, the distinction being not important at this moment), which is coarsely closed under taking subsegments (Theorem B below), and which satisfies coarsely a weak form of $CAT(0)$ condition (Theorem C below).

Recall that Januszkiewicz and Świątkowski [16] considered a system of directed geodesics in a systolic complex (c.f. Definition 2.11). One may try to define the boundary of a systolic complex by taking the inverse limit of the following system. Consider the sequence of combinatorial spheres around a fixed vertex $O$ and projections from larger to smaller spheres along the directed geodesics terminating at $O$. Unfortunately, the inverse limit of this system does not have, in general, property 3 from Theorem A. Property 3 fails, for example, already for the flat systolic plane (c.f. Definition 7.1).

Hence, instead of using directed geodesics, we introduce *Euclidean geodesics*, which behave like $CAT(0)$ geodesics with respect to the flat subcomplexes a systolic complex. To define the Euclidean geodesic between two vertices, say $s, t$, in a systolic complex, we consider the loop obtained by concatenating the two directed geodesics joining $s$ to $t$ and $t$ to $s$. Then we span a minimal surface $S$ on this loop. (Here Elsner's minimal surfaces theory [11] comes in handy. To obtain some uniqueness properties on $S$ we complement Elsner's theory with our results on *layers*, which span the union of all 1–skeleton geodesics between $t$ and $s$.) The surface $S$ is isometric to a contractible subcomplex of the flat systolic plane and hence has a natural structure of a $CAT(0)$ space. The Euclidean geodesic is defined as a sequence of simplices in $S$, which runs near the $CAT(0)$ geodesic between $s$ and $t$.

Now we pass to the more technical part of the exposition. Formally, the Euclidean geodesic is defined for a pair of simplices $\sigma, \tau$ in a systolic complex satisfying $\sigma \subset S_n(\tau), \tau \subset S_n(\sigma)$ for some $n \geq 0$ (where $S_n(\sigma)$ denotes the combinatorial sphere of radius $n$ around $\sigma$, c.f. Definition 2.4). The Euclidean geodesic is a certain sequence of simplices $\delta_k$, where $0 \leq k \leq n$, such that $\delta_0 = \sigma, \delta_n = \tau$, which satisfies $\delta_k \subset S_1(\delta_{k+1}), \delta_{k+1} \subset S_1(\delta_k)$ for $0 \leq k < n$ (c.f. Lemma 9.15(i)). The two most significant features of Euclidean geodesics are given by the following.

**Theorem B** (Theorem 12.2)**.** *Let $\sigma, \tau$ be simplices of a systolic complex $X$, such that for some natural $n$ we have $\sigma \subset S_n(\tau), \tau \subset S_n(\sigma)$. Let $(\delta_k)_{k=0}^n$ be the Euclidean geodesic between $\sigma$ and $\tau$. Take some $0 \leq l < m \leq n$ and let $(r_k)_{k=l}^m$ be a 1–skeleton geodesic such that $r_k \in \delta_k$ for $l \leq k \leq m$. Consider the simplices $\tilde{\delta}_l = r_l, \tilde{\delta}_{l+1}, \ldots, \tilde{\delta}_m = r_m$ of the Euclidean geodesic between vertices $r_l$ and $r_m$. Then for each $l \leq k \leq m$ we have $|\delta_k, \tilde{\delta}_k| \leq C$, where $C$ is a universal constant.*

**Theorem C** (Theorem 13.1)**.** *Let $s, s', t$ be vertices in a systolic complex $X$ such that $|st| = n, |s't| = n'$. Let $(r_k)_{k=0}^n, (r'_k)_{k=0}^{n'}$ be 1–skeleton geodesics such that $r_k \in \delta_k, r'_k \in \delta'_k$, where $(\delta_k), (\delta'_k)$ are Euclidean geodesics for $t, s$ and for $t, s'$ respectively. Then for all $0 \leq c \leq 1$ we have $|r_{\lfloor cn \rfloor} r'_{\lfloor cn' \rfloor}| \leq c|ss'| + C$, where $C$ is a universal constant.*

The article is organized as follows. It consists of an introductory part (Sections 1–2), the two main parts (Sections 3–6 and Sections 7–13), which can be read independently, and of a concluding Section 14.

In Section 2 we give a brief introduction to systolic complexes.

In the first part, assuming we have defined Euclidean geodesics satisfying Theorem B and Theorem C, we define the boundary: In Section 3 we define the boundary as a set of equivalence classes of good geodesic rays. Then we define topology on the compactification obtained by adjoining the boundary (Section 4) and we show its compactness and finite dimensionality (Section 5). Finally, in Section 6, we prove Theorem A —the main result of the paper.

In the second part of the article we define Euclidean geodesics and establish Theorem B and Theorem C: In Section 7 we recall Elsner's results on minimal surfaces. In Section 8 we study *layers*, whose union contains all geodesics between given vertices. We define Euclidean geodesics in Section 9.

In the next two sections we prove Theorem 10.1 which is a weak version of Theorem B (though with a better constant). Apart from the definitions these sections can be skipped by a hurried reader. We decided to include them since this way of obtaining (the weak version of) Theorem B is straightforward in opposition to the strategy in Section 12, which is designed to obtain Theorem C. In Section 10 we study the position of directed geodesics between two simplices of a given Euclidean geodesic with respect to the minimal surface appearing in its construction. Then we verify Theorem 10.1 in Section 11 by studying $CAT(0)$ geometry of minimal surfaces.

The last two sections are devoted to the proofs of Theorem B and Theorem C: In Section 12 we prove (in a technically cumbersome manner) powerful Proposition 12.1 linked with $CAT(0)$ properties of the triangles, whose two sides are Euclidean geodesics. Proposition 12.1 easily implies Theorem B, but its main application comes in Section 13, where we use it to derive Theorem C.

We conclude with announcing some further results for which we do not provide proofs in Section 14.

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# 2 Systolic complexes

In this section we recall (from [16],[17],[15]) the definition and basic properties of systolic complexes and groups.

**Definition 2.1.** A subcomplex $K$ of a simplicial complex $X$ is called *full* in $X$ if any simplex of $X$ spanned by vertices of $K$ is a simplex of $K$. The *span* of a subcomplex $K \subset X$ is the smallest full subcomplex of $X$ containing $K$. We denote it by $\mathrm{span}(K)$. A simplicial complex $X$ is called *flag* if any set of vertices, which are pairwise connected by edges of $X$, spans a simplex in $X$. A simplicial complex $X$ is called *k–large*, $\infty \geq k \geq 4$, if $X$ is flag and there are no embedded cycles of length $< k$, which are full subcomplexes of $X$ (i.e. $X$ is flag and every simplicial loop of length $< k$ and $\geq 4$ "has a diagonal").

**Definition 2.2.** A simplicial complex $X$ is called *systolic* if it is connected, simply connected and links of all simplices in $X$ are 6–large. A group $\Gamma$ is called *systolic* if it acts cocompactly and properly by simplicial automorphisms on a systolic complex $X$. (*Properly* means $X$ is locally finite and for each compact subcomplex $K \subset X$ the set of $\gamma \in \Gamma$ such that $\gamma(K) \cap K \neq \emptyset$ is finite.)

Recall [16], Proposition 1.4, that systolic complexes are themselves 6–large. In particular they are flag. Moreover, we have the following.

**Theorem 2.3** ([16], Theorem 4.1(1)). *Systolic complexes are contractible.*

Now we briefly treat the definitions and facts concerning convexity.

**Definition 2.4.** For every pair of subcomplexes (usually vertices) $A, B$ in a simplicial complex $X$ denote by $|A, B|$ ($|ab|$ for vertices $a, b$) the combinatorial distance between $A^{(0)}, B^{(0)}$ in $X^{(1)}$, the 1–skeleton of $X$ (i.e. the minimal number of edges in a simplicial path connecting both sets). A subcomplex $K$ of a simplicial complex $X$ is called *3–convex* if it is a full subcomplex of $X$ and for every pair of edges $ab, bc$ such that $a, c \in K$, $|ac| = 2$, we have $b \in K$. A subcomplex $K$ of a systolic complex $X$ is called *convex* if it is connected and links of all simplices in $K$ are 3–convex subcomplexes of links of those simplices in $X$.

In Lemma 7.2 of [16] authors conclude that convex subcomplexes of a systolic complex $X$ are full and 3–convex in $X$, and systolic themselves, hence contractible by Theorem 2.3. The intersection of a family of convex subcomplexes is convex. For a subcomplex $Y \subset X$, $n \geq 0$, the *combinatorial ball* $B_n(Y)$ of radius $n$ around $Y$ is the span of $\{p \in X^{(0)} \colon |p, Y| \leq n\}$. (Similarly $S_n(Y) = \text{span}\{p \in X^{(0)} \colon |p, Y| = n\}$.) If $Y$ is convex (in particular, if $Y$ is a simplex) then $B_n(Y)$ is also convex, as proved in [16], Corollary 7.5. Combining this with previous remarks we record:

**Corollary 2.5.** *In systolic complexes, balls around simplices are contractible.*

Haglund–Świątkowski prove the following.

**Proposition 2.6** ([15], Proposition 4.9). *A full subcomplex $Y$ of a systolic complex $X$ is convex if and only if $Y^{(1)}$ is geodesically convex in $X^{(1)}$ (i.e. if all geodesics in $X^{(1)}$ joining vertices of $Y$ lie in $Y^{(1)}$).*

We record:

**Corollary 2.7.** *In systolic complexes balls around simplices are geodesically convex.*

We will need a crucial "projection lemma". The *residue* of a simplex $\sigma$ in $X$ is the union of all simplices in $X$, which contain $\sigma$.

**Lemma 2.8** ([16], Lemma 7.7). *Let $Y$ be a convex subcomplex of a systolic complex $X$ and let $\sigma$ be a simplex in $S_1(Y)$. Then the intersection of the residue of $\sigma$ and of the complex $Y$ is a simplex (in particular it is nonempty).*

**Definition 2.9.** The simplex as in Lemma 2.8 is called the *projection* of $\sigma$ onto $Y$.

The following lemma immediately follows from Definition 2.9.

**Lemma 2.10.** *Let $\sigma \subset \tilde{\sigma}$ be simplices in $S_1(Y)$ for some convex $Y$ and let $\pi, \tilde{\pi}$ be their projections onto $Y$. Then $\tilde{\pi} \subset \pi$.*

**Definition 2.11.** For a pair of vertices $v, w$, $|vw| = n$ in a systolic complex $X$ we define inductively a sequence of simplices $\sigma_0 = v, \sigma_1, \ldots, \sigma_n = w$ as follows. Take $\sigma_i$ equal to the projection of $\sigma_{i-1}$ onto $B_{n-i}(w)$ for $i = 1, \ldots, n-1, n$. The sequence $(\sigma_i)_{i=0}^n$ is called the *directed geodesic* from $v$ to $w$ (this notion is introduced and studied in [16]).

We can extend this construction to any pair $(\sigma_0, W)$, where $W$ is a convex subcomplex of $X$ and $\sigma_0$ is a simplex. Namely, if for some $n$ we have $\sigma_0 \subset S_n(W)$ then take $\sigma_i$ to be the projection of $\sigma_{i-1}$ onto $B_{n-i}(W)$. If $\sigma_0$ intersects both $S_n(W)$ and $S_{n-1}(W)$ then take $\sigma_1 = \sigma_0 \cap S_{n-1}(W)$ and then proceed as previously. We call the final $\sigma_n \subset W$ the *projection* of $\sigma_0$ onto $W$. Note that this coincides with Definition 2.9. Observe that if $\sigma_0 \subset W$ then the projection of $\sigma_0$ onto $W$ is equal to $\sigma_0$.

Finally, recall a powerful observation.

**Lemma 2.12** ([17], Lemma 4.4)**.** *Every full subcomplex of a systolic complex is aspherical.*

# 3 Definition of the boundary

Let $X$ be a systolic complex. In this section we give two equivalent definitions of the boundary of $X$ as a set. We use the notion of Euclidean geodesics which will be introduced in Section 9, but actually we need only its features given by Theorem B and Theorem C. Thus, it is enough to read Sections 1–2 to follow the first part of the article (Sections 3–6). Let $C$ be a natural number, which is a universal constant satisfying assertions of both Theorem B and Theorem C.

**Remark 3.1.** Let $(\delta_i)_{i=0}^n$ be a Euclidean geodesic and let $v_k$ be a vertex in $\delta_k$ for some $0 \leq k \leq n$. Then there exists a 1–skeleton geodesic $(v_i)_{i=0}^n$ such that $v_i \in \delta_i$ for $0 \leq i \leq n$. This follows from the fact that $\delta_{i+1} \subset S_1(\delta_i)$, which we use for $k \leq i < n-1$, and from $\delta_i \subset S_1(\delta_{i+1})$, which we use for $1 \leq i < k$ (see Section 1 or Lemma 9.15(i)).

**Definition 3.2.** Let $v, w$ be vertices of a systolic complex $X$. Let $\gamma = (v_0 = v, v_1, v_2, \ldots, v_n = w)$ be a geodesic in the 1–skeleton of $X$ between $v$ and $w$ or let $\gamma = (v = v_0, v_1, v_2, \ldots)$ be a 1–skeleton geodesic ray starting at $v$ (then we set $n = \infty$). For $0 \le i < j \le n$, by $(\delta_i^{i,j} = v_i, \delta_{i+1}^{i,j}, \dots, \delta_j^{i,j} = v_j)$ we denote the Euclidean geodesic between $v_i$ and $v_j$. We say that $\gamma$ is a *good geodesic* between $v$ and $w$ or that $\gamma$ is a *good geodesic ray* starting at $v$ if for every $0 \le i < j \le n$ and every $i \le k \le j$ we have $|v_k, \delta_k^{i,j}| \le C + 1$ (the constant $C$ is defined in the beginning of this section).

By $\mathcal{R}$ we denote the set of all good geodesic rays in $X$. For a given vertex $O$ of $X$, by $\mathcal{R}_O$ we denote the set of all good geodesic rays starting at $O$.

The following two results are immediate corollaries of Theorem B and Theorem C.

**Corollary 3.3.** *For every two vertices $v, w \in X$ there exists a good geodesic between them.*

**Proof.** Let $(\delta_0 = v, \delta_1, \dots, \delta_n = w)$ be the Euclidean geodesic between $v$ and $w$. By Remark 3.1, there exists a 1–skeleton geodesic $\gamma = (v_0 = v, v_1, v_2, \dots, v_n = w)$ with $v_i \in \delta_i$. We claim that $\gamma$ is a good geodesic. To justify the claim let $0 \le i < j \le n$. Let $(\widetilde{\delta}_i, \widetilde{\delta}_{i+1}, \dots, \widetilde{\delta}_j)$ be the Euclidean geodesic between $v_i$ and $v_j$. By Theorem B, for every $i \le k \le j$, we have

$$|v_k, \widetilde{\delta}_k| \le |\delta_k, \widetilde{\delta}_k| + 1 \le C + 1,$$

which justifies the claim. □

**Corollary 3.4.** *Let $(v_0 = O, v_1, v_2, \dots, v_n)$, $(w_0 = O, w_1, w_2, \dots, w_m)$ be good geodesics in $X$. Then for all $0 \le c \le 1$ we have $|v_{\lfloor cn \rfloor} w_{\lfloor cm \rfloor}| \le c|v_n w_m| + D$, where $D = 3C + 2$.*

**Proof.** Let $(\delta_i^v), (\delta_i^w)$ be the Euclidean geodesics between $O$ and, respectively, $v_n$, $w_m$. Fix $0 \le c \le 1$. Pick vertices $v'_{\lfloor cn \rfloor} \in \delta^v_{\lfloor cn \rfloor}$ and $w'_{\lfloor cm \rfloor} \in \delta^w_{\lfloor cm \rfloor}$ which realize the distance to $v_{\lfloor cn \rfloor}, w_{\lfloor cm \rfloor}$, respectively. Find 1–skeleton geodesics $(v'_i)_{i=0}^{\lfloor cn \rfloor}$ and $(w'_i)_{i=0}^{\lfloor cm \rfloor}$ such that $v'_i \in \delta_i^v$ and $w'_i \in \delta_i^w$. Their existence is guaranteed by Remark 3.1. By Theorem C, we have

$$\begin{aligned}|v_{\lfloor cn \rfloor} w_{\lfloor cm \rfloor}| &\le |v_{\lfloor cn \rfloor} v'_{\lfloor cn \rfloor}| + |v'_{\lfloor cn \rfloor} w'_{\lfloor cm \rfloor}| + |w'_{\lfloor cm \rfloor} w_{\lfloor cm \rfloor}| = \\ &= |v_{\lfloor cn \rfloor}, \delta^v_{\lfloor cn \rfloor}| + |v'_{\lfloor cn \rfloor} w'_{\lfloor cm \rfloor}| + |\delta^w_{\lfloor cm \rfloor}, w_{\lfloor cm \rfloor}| \le \\ &\le (C+1) + (c|v_n w_m| + C) + (C+1),\end{aligned}$$

as desired. □

The following simple corollary of Corollary 3.4 will be useful.

**Corollary 3.5.** *Let* $(v_0 = O, v_1, v_2, \ldots, v_k), (w_0 = O, w_1, w_2, \ldots, w_l)$ *be good geodesics in* $X$*. Then for all* $0 \le N \le \min\{k, l\}$ *we have* $|v_N w_N| \le 2|v_k w_l| + D$*.*

**Proof.** W.l.o.g. we can assume that $k \le l$. Observe that $l - k \le |v_k w_l|$. Hence, by Corollary 3.4, we have

$$|v_N w_N| \le |v_k w_k| + D \le |v_k w_l| + |w_l w_k| + D = |v_k w_l| + (l-k) + D \le 2|v_k w_l| + D.$$

□

Below we define the central object of the article.

**Definition 3.6.** The *(ideal) boundary* of a systolic complex $X$ is the set $\partial X = \mathcal{R}/\sim$ of equivalence classes of good geodesic rays, where rays $\eta = (v_0, v_1, v_2, \ldots)$, $\xi = (w_0, w_1, w_2, \ldots)$ are identified if $|v_i w_i|$ is bounded above by a constant independent of $i$ (one can check this happens exactly when the Hausdorff distance between $\eta$ and $\xi$ is finite). For a good geodesic ray $\eta$, we denote its equivalence class in $\partial X$ by $[\eta]$.

In order to introduce topology on $\overline{X} = X \cup \partial X$ we give another definition of the boundary. The two definitions will turn out to be equivalent in the case of a systolic complex with a geometric group action.

**Definition 3.7.** Let $O$ be a vertex of a systolic complex $X$. Then the *(ideal) boundary of* $X$ *with respect to the basepoint vertex* $O$ is the set $\partial_O X = \mathcal{R}_O/\sim$ of equivalence classes of good geodesic rays starting at $O$, where rays $\eta = (v_0 = O, v_1, v_2, \ldots)$, $\xi = (w_0 = O, w_1, w_2, \ldots)$ are identified if $|v_i w_i|$ is bounded above by a constant independent of $i$ (again this happens exactly when the Hausdorff distance between $\eta$ and $\xi$ is finite). For $\eta \in \mathcal{R}_O$, we denote its equivalence class in $\partial_O X$ by $[\eta]$ (we hope this ambiguity of the notation will not cause confusion).

**Lemma 3.8.** *Let* $\eta = (v_0 = O, v_1, v_2, \ldots)$, $\xi = (w_0 = O, w_1, w_2, \ldots) \in \mathcal{R}_O$*. Then* $[\eta] = [\xi]$ *iff* $|v_i w_i| \le D$ *for all* $i$*.*

**Proof.** We show that if for some $i$ we have $|v_i w_i| - D \ge 1$, then $[\xi] \ne [\eta]$. Let $i$ be as above and $R$ be a natural number. Then, by Corollary 3.4, we have

$$|v_{Ri} w_{Ri}| \ge R(|v_i w_i| - D) \ge R.$$

Since $R$ can be chosen arbitrarily large, we get $[\xi] \ne [\eta]$. □

In the remaining part of this section we prove equivalence of the above two notions of boundaries in the case of locally finite complexes. Assume that $X$ is a locally finite systolic complex. Let $O \in X$ be a fixed vertex and let $\eta = (v^0, v^1, v^2, \ldots)$ be a good geodesic ray in $X$. For every $i \geq 0$ we choose a good geodesic $\eta^i = (v^i_0 = O, v^i_1, v^i_2, \ldots, v^i_{n(i)} = v^i)$, guaranteed by Corollary 3.3. Since $B_1(O)$ is finite, for some vertex $v_1 \in S_1(O)$ there are infinitely many $i$ such that $n(i) = |Ov^i| \geq 1$ and $v^i_1 = v_1$. Similarly, since all balls are finite, we obtain inductively vertices $v_k \in S_k(O)$ satisfying the following. For each $k$ there are infinitely many $i$ such that $n(i) \geq k$ and for all $j \leq k$ we have $v^i_j = v_j$. For each $k$ denote some such $i$ by $i(k)$. The following easy facts hold.

**Lemma 3.9.** *The sequence $(v_0 = O, v_1, v_2, \ldots)$ obtained as above is a good geodesic ray. Moreover, for every $j$ we have $|v^j v_j| \leq 3|Ov^0| + D$.*

**Proof.** The first assertion follows from the fact that for every $k$ the sequence $(v_0 = O, v_1, v_2, \ldots, v_k)$ is a subsequence of the good geodesic $\eta^{i(k)}$ and hence, by Definition 3.2, it is a good geodesic.

Now we prove the second assertion. Let $j \geq 0$. Consider the case $n(i(j)) \leq i(j)$ (the case $n(i(j)) > i(j)$ can be examined analogically). Then for $k = i(j) - n(i(j))$ we have $|v^k v^{i(j)}| = |v^{i(j)}_0 v^{i(j)}|$. Thus we can apply Corollary 3.4 with $m = n$ to good geodesics $\eta^{i(j)}$ and $(v^k, v^{k+1}, \ldots, v^{i(j)})$, which yields the following.

$$|v^{k+j} v_j| = |v^{k+j} v^{i(j)}_j| \leq |v^k v^{i(j)}_0| + D \leq (|Ov^0| + k) + D.$$

Hence

$$|v^j v_j| \leq k + |v^{k+j} v_j| \leq |Ov^0| + 2k + D \leq 3|Ov^0| + D,$$

where the last inequality follows from $k \leq |Ov^0|$, which is the triangle inequality for $v^0, v^{i(j)}$ and $O$. □

**Corollary 3.10.** *Let $X$ be a locally finite systolic complex and $O, O'$ its vertices. Then the map $\Phi_O\colon \partial X \to \partial_O X$ given by $\Phi_O([(v^0, v^1, v^2, \ldots)]) = [(v_0 = O, v_1, v_2, \ldots)]$ is well defined. It is a bijection between $\partial X$ and $\partial_O X$. Its restriction $\Phi_{O'O} = \Phi_O|_{\partial_{O'} X}$ is a bijection between $\partial_{O'} X$ and $\partial_O X$.*

# 4 Topology on $\overline{X} = X \cup \partial_O X$

Let $X$ be a systolic complex and $O \in X$ be its vertex. In this section we define the topology on the set $\overline{X} = X \cup \partial_O X$, which extends the usual topology on the simplicial complex $X$. The idea is to define the topology through neighborhoods (not necessarily open) of points in $\overline{X}$. The only problem is to define the neighborhoods of points in the boundary.

For a 1–skeleton geodesic or a geodesic ray $\eta = (v_0, v_1, v_2, \ldots)$, we denote by $B_1(\eta)$ the combinatorial ball of radius 1 around the subcomplex $\{v_0, v_1, v_2, \ldots\}$. Let $C$ and $D = 3C + 2$ be the constants from the previous section.

**Definition 4.1.** Let $\eta = (v_0 = O, v_1, v_2, \ldots)$ be a good geodesic ray in $X$ and let $R > D$ (i.e. $R \geq D + 1$) and $N \geq 1$ be natural numbers (in fact we could also allow $N = 0$, but this would complicate some computations later on). By $\mathcal{G}_O(\eta, N, R)$ we denote the set of all good geodesics $(w_0 = O, w_1, \ldots, w_k)$ with $k \geq N$ and good geodesic rays $(O = w_0, w_1, \ldots)$, such that $|w_N v_N| \leq R$.

By $\mathcal{G}'_O(\eta, N, R)$ we denote the set $\{(w_N, w_{N+1}, \ldots)|\ (w_0 = O, w_1, \ldots) \in \mathcal{G}_O(\eta, N, R)\}$.

A *standard neighborhood* of $[\eta] \in \partial_O X \subset \overline{X}$ is the set

$$U_O(\eta, N, R) = \{[\xi]|\ \xi \in \mathcal{G}_O(\eta, N, R) \cap \mathcal{R}_O\} \cup \bigcup \{\mathrm{int} B_1(\xi)|\ \xi \in \mathcal{G}'_O(\eta, N, R)\}\,.$$

We write $\mathcal{G}(\eta, N, R)$, $\mathcal{G}'(\eta, N, R)$ and $U(\eta, N, R)$ instead of $\mathcal{G}_O(\eta, N, R)$, $\mathcal{G}'_O(\eta, N, R)$ and $U_O(\eta, N, R)$ if it is clear what is the basepoint $O$.

Before we define the topology, we need the following useful lemmas. The first one is an immediate consequence of the above definition.

**Lemma 4.2.** *Let $\eta, \xi \in \mathcal{R}_O$ and let $N, N', R > D, R' > D$ be natural numbers such that $N' \geq N$. If $\mathcal{G}(\xi, N', R') \subset \mathcal{G}(\eta, N, R)$ then $U(\xi, N', R') \subset U(\eta, N, R)$.*

**Lemma 4.3.** *Let $U(\eta, N, R)$ be a standard neighborhood, let $\xi \in \mathcal{R}_O$ be such that $[\xi] = [\eta]$ and let $R' \geq D$ be a natural number. Then, for $N' \geq (R' + D)N$, we have $U(\xi, N', R') \subset U(\eta, N, R)$.*

**Proof.** Denote $\eta = (v_0 = O, v_1, v_2, \ldots)$ and $\xi = (w_0 = O, w_1, w_2, \ldots)$.

By Lemma 4.2, it is enough to show that for every $\zeta \in \mathcal{G}(\xi, N', R')$ we have $\zeta \in \mathcal{G}(\eta, N, R)$.

Let $\zeta = (z_0 = O, z_1, z_2, \ldots) \in \mathcal{G}(\xi, N', R')$. By Corollary 3.4 and Lemma 3.8, we have

$$\begin{aligned} |z_N v_N| &\leq \frac{1}{R' + D} |z_{N'} v_{N'}| + D \leq \frac{1}{R' + D} (|z_{N'} w_{N'}| + |w_{N'} v_{N'}|) + D \leq \\ &\leq \frac{1}{R' + D} (R' + D) + D \leq R. \end{aligned}$$

Thus $\zeta \in \mathcal{G}(\eta, N, R)$ and the lemma follows. □

The following defines topology on $\overline{X}$.

**Proposition 4.4.** *Let $\mathcal{A}$ be the family of subsets $A$ of $\overline{X} = X \cup \partial_O X$ satisfying the following. $A \cap X$ is open in $X$ and for every $x \in A \cap \partial_O X$ there is some $\eta \in \mathcal{R}_O$ such that $[\eta] = x$ and there is a standard neighborhood $U(\eta, N, R) \subset A$. Then $\mathcal{A}$ is a topology on $\overline{X}$.*

**Proof.** The only thing we have to check is the following. If $A_1, A_2 \in \mathcal{A}$ and $[\eta] \in A_1 \cap A_2 \cap \partial_O X$ then there is a standard neighborhood $U(\eta, N, R)$ of $[\eta]$ contained in $A_1 \cap A_2$.

Since $A_1, A_2 \in \mathcal{A}$, for $i = 1, 2$, there are standard neighborhoods $U(\eta_i, N_i, R_i) \subset A_i$ such that $[\eta_i] = [\eta]$. Thus, by Lemma 4.3, for any natural $R > D$ there exists $N \geq N_i$, $i = 1, 2$, such that $U(\eta, N, R) \subset U(\eta_1, N_1, R_1) \cap U(\eta_2, N_2, R_2) \subset A_1 \cap A_2$. □

**Remark 4.5.** It is easy to verify that when $X$ is $\delta$–hyperbolic (in the sense of Gromov) then our boundary $\partial_O X$ (with topology induced from $\overline{X}$) is homeomorphic in a natural way with the Gromov boundary of $X$.

We still did not prove that the topology defined in Proposition 4.4 is non–trivial. This will follow from the next two lemmas, in which we characterize the intersections with the boundary of the interiors of standard neighborhoods. In particular, we show that $[\xi]$ is contained in the interior of $U(\xi, N, R)$.

**Lemma 4.6.** *For a set $A \subset \overline{X}$, the intersection $\mathrm{int}A \cap \partial_O X$ consists of those points $x \in \partial_O X$ for which there exists a representative $\eta$ with a standard neighborhood $U(\eta, N, R) \subset A$.*

**Proof.** Let $B$ be the set of those points $x \in \partial_O X$ for which there exists a representative $\eta$ of $x$ with a standard neighborhood $U(\eta, N, R) \subset A$.

It is clear that $\mathrm{int}A \cap \partial_O X \subset B$, since $\mathrm{int}A$ is open in the topology defined in Proposition 4.4. We want now to prove the converse inclusion $B \subset \mathrm{int}A \cap \partial_O X$. It is clear that $B \subset A \cap \partial_O X$. Thus to prove the lemma we only have to show that $B$ is open in $\partial_O X$ (in the topology induced from $\overline{X}$).

Let $x \in B$ and let its representative $\eta$ be such that the standard neighborhood $U(\eta, N, R')$ is contained in $A$. By Lemma 4.3, we can assume that $R' \geq 2(D+1)$. Choose natural number $N' \geq RN$. We claim that $U(\eta, N', R) \cap \partial_O X \subset B$ (i.e. that equivalence classes of elements in $\mathcal{G}(\eta, N', R) \cap \mathcal{R}_O$ lie in $B$). This implies that $B$ is open in $\partial_O X$.

To justify the claim let $\xi \in \mathcal{G}(\eta, N', R) \cap \mathcal{R}_O$. To prove that $[\xi] \in B$ it is enough to establish $U(\xi, N', R) \subset U(\eta, N, R')$, since the latter is contained in $A$. By Lemma 4.2, it is enough to show that for every $\zeta \in \mathcal{G}(\xi, N', R)$, we have $\zeta \in \mathcal{G}(\eta, N, R')$. Let $\zeta = (z_0 = O, z_1, \ldots) \in \mathcal{G}(\xi, N', R)$. Denote $\eta = (v_0 = O, v_1, \ldots)$, $\xi = (w_0 = O, w_1, \ldots)$.

By Corollary 3.4, we have

$$\begin{aligned} |z_N v_N| &\leq |z_N w_N| + |w_N v_N| \leq \\ &\leq \Big(\frac{1}{R}|z_{N'} w_{N'}| + D\Big) + \Big(\frac{1}{R}|w_{N'} v_{N'}| + D\Big) \leq \\ &\leq \Big(\frac{1}{R}R + D\Big) + \Big(\frac{1}{R}R + D\Big) = 2(D+1) \leq R'. \end{aligned}$$

Thus $\zeta \in \mathcal{G}(\eta, N, R')$ and it follows that $U(\xi, N', R) \subset U(\eta, N, R')$, which justifies the claim. □

**Lemma 4.7.** *Let $U(\eta, N, R)$ be a standard neighborhood. Let $\xi = (w_0 = O, w_1, w_2, \ldots) \in \mathcal{R}_O$ be such that $v_N = w_N$, where $\eta = (v_0 = O, v_1, v_2, \ldots)$. Then $[\xi]$ is contained in the interior of $U(\eta, N, R)$.*

**Proof.** By Lemma 4.6, it is enough to show that there exists a standard neighborhood $U(\xi, N', R)$ of $[\xi]$ contained in $U(\eta, N, R)$. Let $N' \geq RN$. By Lemma 4.2, it is enough to show that for $(z_0 = O, z_1, z_2, \ldots) \in \mathcal{G}(\xi, N', R)$, we have $|z_N v_N| \leq R$. By Corollary 3.4, we have

$$|z_N v_N| = |z_N w_N| \leq \frac{1}{R}|z_{N'} w_{N'}| + D \leq \frac{1}{R}R + D \leq R,$$

as desired. □

Below we give a sufficient condition for two standard neighborhoods to be disjoint.

**Lemma 4.8.** *Let $U(\eta, N, R)$ and $U(\xi, N, S)$ be two standard neighborhoods, with $\eta = (v_0 = O, v_1, v_2, \ldots)$ and $\xi = (w_0 = O, w_1, w_2, \ldots)$. If $|v_N w_N| > R + S + D + 2$, then $U(\eta, N, R) \cap U(\xi, N, S) = \emptyset$.*

**Proof.** By contradiction. Assume $U(\eta, N, R) \cap U(\xi, N, S) \neq \emptyset$.

**Case 1:** Let $x \in U(\eta, N, R) \cap U(\xi, N, S) \cap X$. Then, by Definition 4.1, there exist $\eta' = (v'_0 = O, v'_1, v'_2, \ldots) \in \mathcal{G}(\eta, N, R)$ and $\xi' = (w'_0 = O, w'_1, w'_2, \ldots) \in \mathcal{G}(\xi, N, S)$ such that $x$ belongs to the interior of both some simplex with vertex $v'_k$ and some simplex with vertex $w'_l$, for some $k, l \geq N$. Then these simplices coincide and $|v'_k w'_l| \leq 1$. By Corollary 3.5, we have

$$|v_N w_N| \leq |v_N v'_N| + |v'_N w'_N| + |w'_N w_N| \leq R + (2|v'_k w'_l| + D) + S \leq R + (2 + D) + S,$$

contradiction.

**Case 2:** Let $\eta' = (v'_0 = O, v'_1, v'_2, \ldots) \in \mathcal{G}(\eta, N, R)$ and $\xi' = (w'_0 = O, w'_1, w'_2, \ldots) \in \mathcal{G}(\xi, N, S)$ be such that $[\eta'] = [\xi']$. Then, by Lemma 3.8, we get

$$|v_N w_N| \leq |v_N v'_N| + |v'_N w'_N| + |w'_N w_N| \leq R + D + S,$$

contradiction. □

# 5 Compactness and finite dimensionality

Let $X$ be a locally finite systolic complex and let $O \in X$ be its vertex. In this section we show that $\overline{X} = X \cup \partial_O X$ is compact metrizable and (if $X$ satisfies some additional local finiteness conditions) finitely dimensional. We also prove that, for a different vertex $O'$ of $X$, the compactifications $X \cup \partial_O X$ and $X \cup \partial_{O'} X$ are homeomorphic.

**Proposition 5.1.** *If $X$ is locally finite then the space $\overline{X} = X \cup \partial_O X$ is second countable and regular.*

**Proof.** It is clear that $\overline{X}$ is second countable. We show that $\overline{X}$ is regular.

First we show that $\overline{X}$ is Hausdorff. We consider only the case of two points of the boundary—the other cases are obvious. Let $[\eta] \neq [\xi]$ be two boundary points with $\eta = (v_0 = O, v_1, v_2, \ldots)$ and $\xi = (w_0 = O, w_1, w_2, \ldots)$. Fix a natural number $R > D$ (for example $R = D + 1$). We can find $N$ such that $|v_N w_N| > 2R + D + 2$. Then, by Lemma 4.7, we have $[\eta] \in \mathrm{int}U(\eta, N, R)$ and $[\xi] \in \mathrm{int}U(\xi, N, R)$ and, by Lemma 4.8, we get $\mathrm{int}U(\eta, N, R) \cap \mathrm{int}U(\xi, N, R) \subset U(\eta, N, R) \cap U(\xi, N, R) = \emptyset$. Thus we get disjoint non–empty open neighborhoods of $[\eta]$ and $[\xi]$.

To show that $\overline{X}$ is regular it is now enough, for every point $x \in \overline{X}$ and every closed subset $A \subset \overline{X}$ which does not contain $x$, to find disjoint open sets $U, V$ such that $x \in U$ and $A \subset V$. Let $x \notin A$ be as above. The case $x \in X$ is obvious thus we consider only the case of $x = [\eta] \in \partial_O X$, for $\eta = (v_0 = O, v_1, v_2, \ldots)$. Fix some natural $R > D$. Since $\overline{X} \setminus A$ is open, by definition of the topology (Proposition 4.4) and by Lemma 4.3, we can find a natural number $N > 0$ such that $U(\eta, N, R') \subset \overline{X} \setminus A$, where $R' \geq 2D + 2$. Let $N' = (R+1)N + 1$ and let $U = \mathrm{int}U(\eta, N', R)$. Observe that, by Lemma 4.7, we have $x \in U$. Now we define $V$. For each $y \in A \cap X$, choose an open set $V_y = \mathrm{int}B_1(z')$ for some vertex $z'$ in $X$ such that $y \in \mathrm{int}B_1(z')$. Then we set $V = \bigcup \{V_y \mid \; y \in A \cap X\} \cup \bigcup \{\mathrm{int}U(\xi, N', R) \mid \; [\xi] \in A \cap \partial_O X\}$. By Lemma 4.7, we have $A \cap \partial_O X \subset V$, hence $A \subset V$. Thus to prove that $U$ and $V$ are as desired we only need to show that $U \cap V = \emptyset$.

First we prove that $U \cap \mathrm{int}U(\xi, N', R) = \emptyset$, for $[\xi] \in A \cap \partial_O X$. Let $\xi = (w_0 = O, w_1, w_2, \ldots)$. Then, by Corollary 3.4 and by $A \cap U(\eta, N, R') = \emptyset$, we have

$$|v_{N'} w_{N'}| \geq \frac{N'}{N}(|v_N w_N| - D) > (R+1)(R' - D) \geq \\ \geq (R+1)(D+2) > 2R + D + 2.$$

Thus, by Lemma 4.8, $U \cap \mathrm{int}U(\xi, N', R) \subset U(\eta, N', R) \cap U(\xi, N', R) = \emptyset$.

Now we show that $U \cap V_y = \emptyset$, for $y \in A \cap X$. By contradiction, assume $p \in U \cap V_y$. Since $p \in U$, there exist a vertex $z$ of the simplex containing $p$ in its interior and a good geodesic $(z_0 = O, z_1, \ldots, z_k = z) \in \mathcal{G}(\eta, N', R)$, where $k \geq N'$. Then, by Corollary 3.4, we have

$$|v_N z_N| \leq \frac{N}{N'}|v_{N'} z_{N'}| + D < \frac{1}{R}R + D \leq D + 1.$$

On the other hand, since $p \in V_y$, there is a vertex $z'$ such that $\{y, p\} \in \mathrm{int}B_1(z')$. Then $|zz'| \leq 1$. Let $(O = z'_0, z'_1, \ldots, z'_l = z')$ be a good geodesic. We have $l \geq N' - 1$, hence by Corollary 3.4 and Corollary 3.5, we get

$$|z_N z'_N| \leq \frac{N}{N'-1}|z_{N'-1} z'_{N'-1}| + D \leq \frac{1}{R+1}(2|zz'| + D) + D \leq \\ \leq \frac{1}{D+2}(2 + D) + D = D + 1.$$

Summarizing, we have $|v_N z'_N| \leq |v_N z_N| + |z_N z'_N| \leq 2D + 2 \leq R'$. It follows that $(O = z'_0, z'_1, \ldots, z'_L = z') \in \mathcal{G}(\eta, N, R')$ and hence $y \in U(\eta, N, R')$—contradiction. □

**Corollary 5.2.** *If $X$ is locally finite then the space $\overline{X} = X \cup \partial_O X$ is metrizable.*

**Proof.** This follows from the Urysohn Metrization Theorem—cf. [10, Corollary 9.2]. □

**Proposition 5.3.** *If $X$ is locally finite then the space $\overline{X} = X \cup \partial_O X$ is compact.*

**Proof.** By Corollary 5.2, it is enough to show that every infinite sequence of points in $\overline{X}$ contains a convergent subsequence. Let $(x^1, x^2, x^3, \ldots)$ be a given sequence of points in $\overline{X}$. If for some $n > 0$ there is only finitely many $x^i$ outside the ball $B_n(O)$ (which is finite), then we can find a convergent subsequence. From now on we assume there is no $n$ as above.

For every $i$ we choose a good geodesic or a good geodesic ray $\eta^i = (v_0^i = O, v_1^i, v_2^i, \ldots)$ the following way. If $x^i \in X$ then $\eta^i = (v_0^i = O, v_1^i, v_2^i, \ldots, v_{n(i)}^i)$ is a good geodesic between $O$ and a vertex $v_{n(i)}^i$ lying in a common simplex with the point $x^i$. If $x^i \in \partial_O X$ then we set $\eta^i = \zeta$ for an arbitrary $\zeta$ such that $x^i = [\zeta]$ and we set $n(i) = \infty$. By our assumptions on $(x^1, x^2, x^3, \ldots)$, for every $n > 0$ there exists an arbitrarily large $i$ such that $n(i) > n$. Since $S_1(O)$ is finite, for some vertex $v_1 \in S_1(O)$ there are infinitely many $i$ such that $n(i) \geq 1$ and $v_1^i = v_1$. Let $i(1)$ be some such $i$. Similarly, since all spheres are finite, we obtain inductively vertices $v_k \in S_k(O)$ and numbers $i(k)$ satisfying the following. For each $k$ there are infinitely many $i$ such that $n(i) \geq k$ and for all $j \leq k$ we have $v_j^i = v_j$; we denote some such $i > i(k-1)$ by $i(k)$.

Observe that for every $k$ the sequence $(v_0 = O, v_1, v_2, \ldots, v_k)$ is a subsequence of the good geodesic or the good geodesic ray $\eta^{i(k)}$ and hence, by Definition 3.2, it is a good geodesic. Thus every subsequence of the infinite sequence $(v_0 = O, v_1, v_2, \ldots)$ is a good geodesic and again, by Definition 3.2, $(v_0 = O, v_1, v_2, \ldots)$ is a good geodesic ray.

We claim that the sequence $(x^{i(k)})_{k=1}^{\infty}$ of points of $\overline{X}$ converges to $[\eta] \in \partial_O X$, where $\eta = (v_0 = O, v_1, v_2, \ldots)$. To prove the claim it is enough to show (since every open set containing $[\eta]$ contains some $U(\eta, N, R)$, by Lemma 4.3) that we have $\eta^{i(k)} \in \mathcal{G}(\eta, N, R)$, for every $k \geq N$. This follows from the equality $v_N^{i(k)} = v_N$, which holds for every $k \geq N$. □

Observe that by the above proof we get the following.

**Corollary 5.4.** *If a locally finite systolic complex is unbounded then its boundary is non–empty.*

Below we prove that the bijection $\Phi_{O'O}$ defined in Corollary 3.10 extends to a homeomorphism of compactifications coming from different basepoints.

**Lemma 5.5.** *Let $X$ be a locally finite systolic complex and let $O, O'$ be its vertices. Then the map $\Phi_{O'O}\colon X \cup \partial_{O'} X \to X \cup \partial_O X$ defined as an extension by identity on $X$ of the map $\Phi_{O'O}\colon \partial_{O'} X \to \partial_O X$ is a homeomorphism.*

**Proof.** By compactness (Proposition 5.3) and by Corollary 3.10, we only have to show that $\Phi_{O'O}$ is continuous. It is enough to check the continuity at the boundary points. Let $\xi = (v_0 = O, v_1, v_2, \ldots)$ be obtained from a good geodesic ray $\eta = (v^0 = O', v^1, v^2, \ldots)$ as in the definition of the map $\Phi_{O'O}$. We show that $\Phi_{O'O}$ is continuous at $[\eta]$. Let $d = |OO'|$, let $R > D$ be a natural number and $R' = R + 3D + 6d$ and let $U$ be an open neighborhood of $[\xi]$ in $X \cup \partial_O X$. We have to show that there exists an open neighborhood

$V$ of $[\eta]$ in $X \cup \partial_{O'} X$ such that $\Phi_{O'O}(V) \subset U$. By Lemma 4.3, there exists $N$ such that $U_O(\xi, N, R') \subset U$. Let $V = \mathrm{int} U_{O'}(\eta, N+d, R)$. By Lemma 4.7, $[\eta] \in V$. We claim that $\Phi_{O'O}(V) \subset U$—this will finish the proof.

First we show that for $x \in V \cap X$ we have $\Phi_{O'O}(x) = x \in U$. For such an $x$ choose, by definition of $U_{O'}(\eta, N+d, R)$, a good geodesic $(w^0 = O', w^1, w^2, \ldots, w^k) \in \mathcal{G}_{O'}(\eta, N+d, R)$ such that $x$ belongs to the interior of a simplex with vertex $w^k$, where $k \geq N+d$. Let $\zeta = (w_0 = O, w_1, w_2, \ldots, w_l = w^k)$ be a good geodesic guaranteed by Corollary 3.3. Then $|l-k| \leq d$, hence $l \geq N$ and $w_N$ is defined. By Lemma 3.9 and Corollary 3.4, we have

$$\begin{aligned} |w_N v_N| &\leq |w_N w^N| + |w^N v^N| + |v^N v_N| \leq \\ &\leq (3d + D) + (|w^{N+d} v^{N+d}| + D) + (3d + D) \leq \\ &\leq R + 3D + 6d = R'. \end{aligned}$$

This inequality implies that $\zeta \in \mathcal{G}_O(\xi, N, R')$ and hence $x \in U_O(\xi, N, R') \subset U$.

Now we show that for $[\rho] \in V \cap \partial_{O'} X$ we have $\Phi_{O'O}([\rho]) \in U$. Let $\rho = (w^0 = O', w^1, w^2, \ldots) \in \mathcal{G}_{O'}(\eta, N+d, R) \cap \mathcal{R}_{O'}$. Let $\zeta = (w_0 = O, w_1, w_2, \ldots)$ be obtained from $\rho$ as in the definition of $\Phi_{O'O}$. Then, by Lemma 3.9 and Corollary 3.4, we can perform the same computation as in the previous case to get $|w_N v_N| \leq R + 3D + 6d = R'$. Thus $\Phi_{O'O}([\rho]) = [\zeta] \in U_O(\xi, N, R') \subset U$ and we have completed the proof of $\Phi_{O'O}(V) \subset U$ and of the whole lemma. □

Now we proceed to the question of finite dimensionality of $\overline{X}$. Let us remind that a simplicial complex $X$ is *uniformly locally finite* if there exists a natural number $L$ such that every vertex belongs to at most $L$ different simplices. This happens for example when some group acts geometrically on $X$.

**Proposition 5.6.** *Let $X$ be a uniformly locally finite systolic complex. Then $\overline{X} = X \cup \partial_O X$ is finitely dimensional.*

**Proof.** Recall that a space $Y$ has *dimension at most $n$* if, for every open cover $\mathcal{U}$ of $Y$, there exists an open cover $\mathcal{V} \prec \mathcal{U}$ ($\mathcal{V}$ is a *refinement* of $\mathcal{U}$, i.e. every element of $\mathcal{V}$ is contained in some element of $\mathcal{U}$) such that every point in $Y$ belongs to at most $n+1$ elements of $\mathcal{V}$ (i.e. the *multiplicity* of $\mathcal{V}$ is at most $n+1$).

It is clear that $X$ is finitely dimensional. It is thus enough to show that there exists a constant $K$ such that for every open (in $\overline{X}$) cover $\mathcal{U}$ of $\partial_O X$ there exists an open cover $\mathcal{V} \prec \mathcal{U}$ of $\partial_O X$ of multiplicity at most $K$.

Let $R > D$ be a natural number. Then, by uniform local finiteness, there is a constant $K$ such that every ball of radius at most $2R + D + 2$ contains at most $K$ vertices.

Let $\mathcal{U}$ be an open cover of $\partial_O X$ in $\overline{X}$. We construct an open cover $\mathcal{V} \prec \mathcal{U}$ of $\partial_O X$ in $\overline{X}$ consisting of interiors of standard neighborhoods such that the multiplicity of $\mathcal{V}$ is at most $K$.

Let $R' = 2R + 2D$. By the definition of topology (Proposition 4.4) and by Lemma 4.3, for every $[\eta] \in \partial_O X$ there exists a standard neighborhood $U(\eta, N_\eta, R')$ contained in some element of $\mathcal{U}$. By Lemma 4.7 we have $[\eta] \in \mathrm{int}U(\eta, N_\eta, R')$. By compactness (Proposition 5.3), among such neighborhoods we can find a finite family $\left\{U(\eta^j, N_{\eta^j}, R')\right\}_{j=1}^m$ such that the family of smaller standard neighborhoods $\left\{U(\eta^j, N_{\eta^j}, R)\right\}_{j=1}^m$ covers $\partial_O X$. Let $N = \max\{N_{\eta^1}, N_{\eta^2}, \ldots, N_{\eta^m}\}$. Let $A$ denote the set of vertices $v$ in $S_N(O)$ for which there exists a good geodesic ray starting at $O$ and passing through $v$. For each $v \in A$, pick some such good geodesic ray $\xi^v = (w_0^v = O, w_1^v, w_2^v, \ldots, w_N^v = v, \ldots)$. We claim that the family $\mathcal{V} = \{\mathrm{int}U(\xi^v, N, R)|\ v \in A\}$ is as desired.

First we show that $\mathcal{V}$ covers $\partial_O X$. Let $\zeta = (z_0 = O, z_1, z_2, \ldots)$ be an arbitrary good geodesic ray. Then $z_N = w_N^{z_N}$ and thus, by Lemma 4.7, $[\zeta] \in \mathrm{int}U(\xi_{z_N}, N, R)$.

Now we show that $\mathcal{V} \prec \mathcal{U}$. To prove this it is enough to show that for every $v \in A$ there exists $j \in \{1, 2, \ldots, m\}$ such that $U(\xi^v, N, R) \subset U(\eta^j, N_{\eta^j}, R')$. Let $v \in A$. Choose $j$ such that $[\xi^v] \in U(\eta^j, N_{\eta^j}, R)$. By Lemma 4.2, to show that $U(\xi^v, N, R) \subset U(\eta^j, N_{\eta^j}, R')$ it is enough to show that, for every $\zeta \in \mathcal{G}(\xi^v, N, R)$, we have $\zeta \in \mathcal{G}(\eta^j, N_{\eta^j}, R')$. Let $\zeta = (z_0 = O, z_1, z_2, \ldots, z_N, \ldots) \in \mathcal{G}(\xi^v, N, R)$ and denote $\eta^j = (v_0^j = O, v_1^j, v_2^j, \ldots)$. By Lemma 3.8, we have $|w_{N_{\eta_j}}^v v_{N_{\eta_j}}^j| \leq R + D$. Then, by Corollary 3.4, we have

$$\begin{aligned}|z_{N_{\eta_j}} v_{N_{\eta_j}}^j| \leq |z_{N_{\eta_j}} w_{N_{\eta_j}}^v| + |w_{N_{\eta_j}}^v v_{N_{\eta_j}}^j| \leq (|z_N w_N^v| + D) + (R + D) \leq \\ \leq 2R + 2D = R'.\end{aligned}$$

Thus $\zeta \in \mathcal{G}(\eta^j, N_{\eta^j}, R')$ and it follows that $\mathcal{V} \prec \mathcal{U}$.

Finally, we claim that the multiplicity of $\mathcal{V}$ is at most $K$. By Lemma 4.8, if $|vv'| > 2R + D + 2$ then $\mathrm{int}U(\xi^v, N, R) \cap \mathrm{int}U(\xi^{v'}, N, R) \subset U(\xi^v, N, R) \cap U(\xi^{v'}, N, R) = \emptyset$. Thus multiplicity of $\mathcal{V}$ is at most the number of vertices in a ball of radius $2R + D + 2$ in $X$, i.e. it is at most $K$. □

# 6 The main result

The aim of this section is to prove the main result of the paper—Theorem A (Theorem 6.3).

The following result will be crucial.

**Proposition 6.1** ([3, Proposition 2.1], [2, Lemma 1.3])**.** *Let $(Y, Z)$ be a pair of finite–dimensional compact metrizable spaces with $Z$ nowhere dense in $Y$, and such that $Y \setminus Z$ is contractible and locally contractible and the following condition holds:*

- *For every $z \in Z$ and every open neighborhood $U$ of $z$ in $Y$, there exists an open neighborhood $V$ of $z$ contained in $U$ such that $V \setminus Z \hookrightarrow U \setminus Z$ is null–homotopic.*

*Then $Y$ is an ER and $Z$ is a $Z$–set in $Y$.*

Before proving Theorem A we need an important preparatory lemma.

**Lemma 6.2.** *Let $[\eta] \in \partial_O X$ and let $U(\eta, N, R)$ be a standard neighborhood of $[\eta]$ in $\overline{X}$. Then there exists $N'$ such that $U(\eta, N', R) \subset U(\eta, N, R)$ and the inclusion map $U(\eta, N', R) \cap X \hookrightarrow U(\eta, N, R) \cap X$ is null–homotopic.*

**Proof.** Let $R' = 4D+7$. By Lemma 4.3, there exists $\widetilde{N}$ such that $U(\eta, \widetilde{N}, R') \subset U(\eta, N, R)$, so that it is enough to prove the following. For natural $R \geq D$ there exists $N'$ such that $U(\eta, N', R) \subset U(\eta, N, R')$ and the inclusion map $U(\eta, N', R) \cap X \hookrightarrow U(\eta, N, R') \cap X$ is null–homotopic.

Before we start, let us give a rough idea of the proof. Let us restrict to the problem of contracting loops from $U(\eta, N', R) \cap X$ in $U(\eta, N, R') \cap X$ (this turns out to be the most complicated case). Let $\alpha$ be such a loop. We connect each vertex of $\alpha$ by a good geodesic with $O$, and we are interested in the vertex of this geodesic at certain distance $M$ from $O$, where $N < M < N'$. All vertices constructed in this way lie in a certain ball (see Condition 1 below), which is in turn contained in $U(\eta, N, R') \cap X$ (see Condition 3 below). If we connect these vertices by 1–skeleton geodesics in the right order we obtain a loop $\alpha_M$, which lies in the ball considered (Corollary 2.7) and is contractible inside this ball (Corollary 2.5). So we need to find a free homotopy between $\alpha$ and $\alpha_M$, which we construct via intermediate loops $\alpha_l$. To find that two such consecutive loops are homotopic in $U(\eta, N, R') \cap X$, we need Condition 2. This condition guarantees that all relatively small loops by which consecutive $\alpha_l$ differ can be contracted inside $U(\eta, N, R') \cap X$.

Let $M = N + R + 1$ and $N' - 1 \geq (R + D + 4)M$. We will show that $N'$ is as desired. Denote $\eta = (v_0 = O, v_1, v_2, \ldots)$. The choice of $M$ and $N'$ guarantees that the following three conditions hold.

**Condition 1.** Let $\xi = (w_0 = O, w_1, \ldots, w_k)$ be a good geodesic with $k \geq N' - 1$ and $w_k \in \overline{U(\eta, N', R) \cap X}$. Then $w_M \in B_{D+1}(v_M)$.

Indeed, let $(z_0 = O, z_1, \ldots, z_l) \in \mathcal{G}(\eta, N', R)$ be such that $|w_k z_l| \leq 1$ (guaranteed by definition of $U(\eta, N', R)$). Since $k \geq N' - 1$, we have, by Corollary 3.5, that

$$\begin{aligned}|w_{N'-1}v_{N'-1}| \leq |w_{N'-1}z_{N'-1}| + |z_{N'-1}v_{N'-1}| \leq \\ \leq (2|w_k z_l| + D) + (1 + |z_{N'}v_{N'}| + 1) \leq R + D + 4.\end{aligned}$$

Thus, by Corollary 3.4, we have

$$|w_M v_M| \leq \frac{M}{N'-1}|w_{N'-1}v_{N'-1}| + D \leq \frac{1}{R+D+4}|w_{N'-1}v_{N'-1}| + D \leq D + 1.$$

**Condition 2.** Let $\xi = (w_0 = O, w_1, \ldots, w_k)$ be as in Condition 1. Then, for every $k \geq l \geq M + 1$ we have $B_{D+3}(w_l) \subset U(\eta, N, R') \cap X$.

To show this observe that, as in the proof of the previous condition, we have $|w_{N'-1}v_{N'-1}| \leq R + D + 4$. Now, let $z$ be a vertex of $B_{D+3}(w_l)$. Choose a good geodesic $(z_0 = O, z_1, z_2, \ldots, z_m = z)$ (guaranteed by Corollary 3.3). Since $l \geq M + 1 = N + R + 2 \geq N + (D + 3)$, we have that $m \geq N$ and $z_N$ is defined. Thus, by Corollary 3.4 and Corollary 3.5, we have

$$\begin{aligned}|z_N v_N| \leq |z_N w_N| + |w_N v_N| \leq (2|z_m w_l| + D) + \Big(\frac{N}{N'-1}|w_{N'-1}v_{N'-1}| + D\Big) < \\ < (2(D+3) + D) + \Big(\frac{1}{R+D+4}(R + D + 4) + D\Big) = 4D + 7 = R'.\end{aligned}$$

Thus $z \in U(\eta, N, R') \cap X$ and it follows that $B_{D+3}(w_l) \subset U(\eta, N, R') \cap X$.

**Condition 3.** We have $B_{D+1}(v_M) \subset U(\eta, N, R') \cap X$.

This follows immediately from Condition 2, but we want to record it separately.

**The goal.** First observe that $U(\eta, N', R) \subset U(\eta, N, R')$ by Lemma 4.3 and the definition of $N'$. Now we show that the map $\pi_i(U(\eta, N', R) \cap X) \to \pi_i(U(\eta, N, R') \cap X)$ induced by inclusion is trivial, for every $i = 0, 1, 2, \ldots$. Let $A$ be the smallest full subcomplex of $X$ containing $U(\eta, N', R) \cap X$. Observe that the vertices of $A$ lie in $\overline{U(\eta, N', R) \cap X}$. By Condition 2, $A$

is contained in $U(\eta, N, R') \cap X$. Thus it is enough to show that the map $\pi_i(A) \to \pi_i(U(\eta, N, R') \cap X)$ induced by the inclusion is trivial and we may restrict ourselves only to simplicial spherical cycles.

**Case ($i = 0$).** Let $z^1, z^2$ be two vertices of $A$. We will construct a simplicial path in $U(\eta, N, R') \cap X$ connecting $z^1$ and $z^2$.

Choose (using Corollary 3.3) good geodesics $(z_0^j = O, z_1^j, \ldots, z_{k(j)}^j = z^j)$, $j = 1, 2$. By Condition 2, $(z_M^j, z_{M+1}^j, \ldots, z_{k(j)}^j = z^j)$ is contained in $U(\eta, N, R')$ and, by Condition 1, we have $z_M^j \in B_{D+1}(v_M)$. Choose a 1–skeleton geodesic $(u_1 = z_M^1, u_2, \ldots, u_l = z_M^2)$. Since balls are geodesically convex (Corollary 2.7), this geodesic is contained in $B_{D+1}(v_M)$ and hence, by Condition 3, it is contained in $U(\eta, N, R') \cap X$.

Then the 1–skeleton path $(z^1 = z_{k(1)}^1, z_{k(1)-1}^1, \ldots, z_M^1 = u_1, u_2, \ldots, u_l = z_M^2, z_{M+1}^2, \ldots, z_{k(2)}^2 = z^2)$ connects $z^1$ and $z^2$ and is contained in $U(\eta, N, R') \cap X$. Therefore the map $\pi_0(A) \to \pi_0(U(\eta, N, R') \cap X)$ is trivial.

**Case ($i = 1$).** Let $\alpha = (z^0, z^1, \ldots, z^n = z^0)$ be a 1–skeleton loop in $A$. We show that this loop can be contracted within $U(\eta, N, R') \cap X$.

Choose good geodesics $(z_0^j = O, z_1^j, \ldots, z_{k(j)}^j = z^j)$ (guaranteed by Corollary 3.3), for $j = 0, 1, 2, \ldots, n-1$. By $z_k^j$, for $k > k(j)$, we denote $z^j$. Let $K = \max\{k(0), k(1), \ldots, k(n-1)\}$. Observe that, by Corollary 3.5, we have $|z_l^j z_l^{j+1}| \leq D + 2$ (we consider $j$ modulo $n$), for every $l = M, M+1, M+2, \ldots, K$ (we are not interested in smaller $l$). For these $l$ let $(z_l^j = t_l^{j,0}, t_l^{j,1}, \ldots, t_l^{j,p_l(j)} = z_l^{j+1})$ be arbitrary 1–skeleton geodesics. Record that $p_l(j) \leq D + 2$.

Thus, for every $l = M+1, M+2, \ldots, K$ and for every $j = 0, 1, \ldots, n-1$, we have a 1–skeleton loop

$$\gamma_l^j = (z_l^j, z_{l-1}^j = t_{l-1}^{j,0}, t_{l-1}^{j,1}, \ldots, t_{l-1}^{j,p_{l-1}(j)} = z_{l-1}^{j+1}, \\ z_l^{j+1} = t^{j,p_l(j)}, t_l^{j,p_l(j)-1}, \ldots, t^{j,0} = z_l^j)$$

of length at most

$$1 + p_{l-1}(j) + 1 + p_l(j) \leq 1 + (D+2) + 1 + (D+2) = 2D + 6.$$

Hence $\gamma_l^j \subset B_{D+3}(z_l^j)$. Since balls are contractible (Corollary 2.5), $\gamma_l^j$ is contractible inside $B_{D+3}(z_l^j)$, which is, by Condition 2, contained in $U(\eta, N, R')$. Thus, for $M \leq l \leq K$, the loops

$$\alpha_l = (z_l^0 = t_l^{0,0}, t_l^{0,1}, \ldots, t_l^{0,p_l(0)} = z_l^1 = t_l^{1,0}, t_l^{1,1}, \ldots, t_l^{1,p_l(1)} = z_l^2, \ldots \\ \ldots, z_l^{n-1} = t_l^{n-1,0}, t_l^{n-1,1}, \ldots, t_l^{n-1,p_l(n-1)} = z_l^n = z_l^0)$$

for consecutive $l$ are freely homotopic in $U(\eta, N, R')$.

Observe that $\alpha = \alpha_K$. On the other hand $\alpha_M \subset B_{D+1}(v_M)$, by Condition 1 and by geodesic convexity of balls (Corollary 2.7). Moreover, since balls are contractible (Corollary 2.5), $\alpha_M$ can be contracted inside $B_{D+1}(v_M)$, which lies in $U(\eta, N, R')$, by Condition 3. Thus $\alpha$ is contractible in $U(\eta, N, R')$. It follows that the map $\pi_1(A) \to \pi_1(U(\eta, N, R) \cap X)$ is trivial.

**Case ($i > 1$).** Since $A$ is a full subcomplex of a systolic complex it is, by Lemma 2.12, aspherical and thus $\pi_i(A) = 0$ and the map in question is obviously trivial. □

**Theorem 6.3** (Theorem A)**.** *Let a group $G$ act geometrically by simplicial automorphisms on a systolic complex $X$. Then $\overline{X} = X \cup \partial_O X$, where $O$ is a vertex of $X$, is a compactification of $X$ satisfying the following:*

1. *$\overline{X}$ is a Euclidean retract (ER),*
2. *$\partial_O X$ is a $Z$–set in $\overline{X}$,*
3. *for every compact set $K \subset X$, $(gK)_{g\in G}$ is a null sequence,*
4. *the action of $G$ on $X$ extends to an action, by homeomorphisms, of $G$ on $\overline{X}$.*

**Proof.** *(1. and 2.)* By Corollary 5.2, Proposition 5.3, and Proposition 5.6, we have that $\overline{X} = X \cup \partial_O X$ is a finitely dimensional metrizable compact space.

Since $X$ is a simplicial complex, it is locally contractible and, by Theorem 2.3, it is contractible since it is a systolic complex. By the definition of the topology on $\overline{X}$ (c.f. Proposition 4.4), it is clear that $\partial_O X$ is nowhere dense in $\overline{X}$. Thus we are in a position to apply Proposition 6.1. Let $x \in \partial_O X$ and let $U$ be its open neighborhood in $\overline{X}$.

By definition of the topology (Proposition 4.4) we can find a standard neighborhood $U(\eta, N, R) \subset U$, where $[\eta] = x$. By Lemma 6.2, there exists a standard neighborhood $U(\eta, N', R) \subset U(\eta, N, R) \subset U$ (with $[\eta] \in \mathrm{int}(U(\eta, N', R)$, by Lemma 4.7) such that the map $\mathrm{int}(U(\eta, N', R) \cap X) \hookrightarrow U(\eta, N', R) \cap X \hookrightarrow U(\eta, N, R) \cap X \hookrightarrow U \cap X$ is null–homotopic. Thus $\overline{X}$ is an ER and $\partial_O X$ is a $Z$–set in $\overline{X}$.

*(3.)* Let $\mathcal{U}$ be an open cover of $\overline{X}$ and let $K \subset X$ be a compact set. We will show that all but finitely many translates $gK$, for $g \in G$, are $\mathcal{U}$–small.

Let $R > D$ be such that $K \subset B_R(z)$, for some vertex $z$. As in the proof of Proposition 5.6, we can find a natural number $N$, a finite set of vertices $A \subset S_N(O)$ and a collection of good geodesic rays $\{\xi^v \mid v \in A\}$

with $\xi^v$ passing through $v$ such that the following holds. The family $\mathcal{V} = \{\mathrm{int}U(\xi^v, N, R) \mid \ v \in A\}$ covers $\partial_O X$ and the family $\mathcal{V}' = \{U(\xi^v, N, 4R) \mid \ v \in A\}$ is a refinement of $\mathcal{U}$. Thus we can find an open cover $\mathcal{W} = \mathcal{V} \cup \mathcal{W}'$ of $\overline{X}$ such that every $W \in \mathcal{W}'$ is contained in $X$. By compactness—Proposition 5.3—there is a finite subfamily of $\mathcal{W}$ covering $\overline{X}$. It follows that there exists a natural number $N' > N$ such that $\overline{X} \setminus B_{N'}(O) \subset \bigcup \mathcal{V}$. By properness of the action there exists a cofinite subset $H \subset G$ such that $gK \subset B_R(gz) \subset X \setminus B_{N'}(O)$, for $g \in H$.

We claim that, for every $g \in H$, we have $gK \subset B_R(gz) \subset U(\xi^v, N, 4R) \cap X$, for some $v \in A$. Assertion *(3.)* follows then from the claim. Let $g \in H$. Since $\overline{X} \setminus B_{N'}(O) \subset \bigcup \mathcal{V}$, there exists $v \in A$ such that $gz \in \mathrm{int}U(\xi^v, N, R)$. We show that $B_R(gz) \subset U(\xi^v, N, 4R)$. Let $x \in B_R(gz)$ and let $\zeta = (z'_0 = O, z'_1, \ldots, z'_l)$ be a good geodesic (which exists by Corollary 3.3) such that $z'_l \in B_R(gz)$ is a vertex of the simplex containing $x$ in its interior. Since $gz \in U(\xi^v, N, R)$ there exists a good geodesic $(z_0 = O, z_1, z_2, \ldots, z_k = gz)$, such that $|z_N v| \leq R$. We have $l, k \geq N'$ and $|z'_l z_k| \leq R$. Hence, by Corollary 3.5, we have

$$|z'_N v| \leq |z'_N z_N| + |z_N v| \leq (2|z'_l z_k| + D) + |z_N v| \leq \\ \leq (2R + D) + R < 4R.$$

Thus $\zeta \in \mathcal{G}(\xi^v, N, 4R)$ and hence $x \in U(\xi^v, N, 4R)$. It follows that $B_R(gz) \subset U(\xi^v, N, 4R)$. Since $g \in H$ was arbitrary we have that elements of $(gK)_{g \in H}$ are $\mathcal{V}'$–small and thus they are $\mathcal{U}$–small.

*(4.)* For $g \in G$ we define a map $g\circ\colon X \cup \partial_O X \to X \cup \partial_{gO} X$ as follows. For $x \in X$ let $g \circ x = gx$ and for $x = [(v_0 = O, v_1, v_2, \ldots)] \in \partial_O X$ let $g \circ x = [(gv_0 = gO, gv_1, gv_2, \ldots)]$. This is obviously a well defined homeomorphism.

We extend the action of $G$ on $X$ to $X \cup \partial_O X$ by the formula $g \cdot x = \Phi_{gOO}(g \circ x)$, for $x \in \partial_O X$. By Lemma 5.5, the map $g\,\cdot : X \cup \partial_O X \to X \cup \partial_O X$ is a homeomorphism. To see that $(gh) \cdot x = g \cdot (h \cdot x)$, for $x \in \partial_O X$, pick some representative $\eta = (v_0 = O, v_1, \ldots)$ of $x$. We need to show that

$$\Phi_{ghOO}(gh \circ [\eta]) = \Phi_{gOO}(g \circ \Phi_{hOO}(h \circ [\eta])).$$

Recall that, by Lemma 3.9, mappings $\Phi_{gOO}, \Phi_{hOO}$ and $\Phi_{ghOO}$ displace representative rays by a finite Hausdorff distance. Hence $\Phi_{ghOO}(gh \circ [\eta])$ is the class of rays starting at $O$ at a finite Hausdorff distance from $(ghv_0 = ghO, ghv_1, \ldots)$. On the other hand, $\Phi_{hOO}(h \circ [\eta])$ is the class of rays starting at $O$ at a finite Hausdorff distance from $(hv_0 = hO, hv_1, \ldots)$, hence $g \circ \Phi_{hOO}(h \circ [\eta])$ as well as $\Phi_{gOO}(g \circ \Phi_{hOO}(h \circ [\eta]))$ is the class of rays (starting at, respectively, $gO$ and $O$) at a finite Hausdorff distance from $(ghv_0 = ghO, ghv_1, \ldots)$. This proves the desired equality.

Hence we get an extension of the action of $G$ on $X$ to an action on $\overline{X}$ by homeomorphisms. □

# 7 Flat surfaces

With this section we start the second part of the article, in which we define Euclidean geodesics, establish Theorem B and Theorem C. Before we define Euclidean geodesics, we first need to study, as mentioned in Section 1, the minimal surface spanned on a pair of directed geodesics connecting given vertices. The tools for this are minimal surfaces (Section 7) and layers (Section 8).

In this section we recall some definitions and facts concerning flat minimal surfaces in systolic complexes proved by Elsner [11],[12].

**Definition 7.1.** The *flat systolic plane* is a systolic 2–complex obtained by equilaterally triangulating Euclidean plane. We denote it by $\mathbb{E}^2_\Delta$. A *systolic disc* is a systolic triangulation of a 2–disc and a *flat disc* is any systolic disc $\Delta$, which can be embedded into $\mathbb{E}^2_\Delta$, such that $\Delta^{(1)}$ is embedded isometrically into 1–skeleton of $\mathbb{E}^2_\Delta$. A systolic disc $\Delta$ is called *wide* if $\partial\Delta$ is a full subcomplex of $\Delta$. For any vertex $v \in \Delta^{(0)}$ the *defect* at $v$ (denoted by $\mathrm{def}(v)$) is $6 - t(v)$ for $v \notin \partial\Delta^{(0)}$, and $3 - t(v)$ for $v \in \partial\Delta^{(0)}$, where $t(v)$ is the number of triangles in $\Delta$ containing $v$. It is clear that internal vertices of a systolic disc have nonpositive defects.

We will need the following easy and well known fact.

**Lemma 7.2** (Gauss-Bonnet Lemma)**.** *If $\Delta$ is any triangulation of a 2–disc, then*

$$\sum_{v \in \Delta^{(0)}} \mathrm{def}(v) = 6$$

Flat systolic discs can be characterized as follows.

**Lemma 7.3** ([11], Lemma 2.5)**.** *A systolic disc $D$ is flat if and only if it satisfies the following three conditions:*

*(i) $D$ has no internal vertices of defect $< 0$*

*(ii) $D$ has no boundary vertices of defect $< -1$*

*(iii) any segment in $\partial D$ connecting vertices with defect $< 0$ contains a vertex of defect $> 0$.*

Now we recall another handful of definitions.

**Definition 7.4.** Let $X$ be a systolic complex. Any simplicial map $S\colon \Delta \to X$, where $\Delta$ is a triangulation of a 2–disc, is called a *surface.* We say that $S$ is *spanned* on a loop $\gamma$, if $S|_{\partial\Delta} = \gamma$. A loop $\gamma$ is *triangulable*, if there exists a surface $S$ spanned on $\gamma$, such that all the vertices of $\Delta$ are in $\partial\Delta$. A surface $S$ is *systolic, flat* or *wide* if the disc $\Delta$ satisfies the corresponding property. If $S$ is injective on $\partial\Delta$ and minimal (the smallest number of triangles in $\Delta$) among surfaces with the given image of $\partial\Delta$, then $S$ is called *minimal.* A geodesic in $\Delta^{(1)}$ is called *neat* if it stays out of $\partial\Delta$ except possibly at its endpoints. A surface $S$ is called *almost geodesic* if it maps neat geodesics in $\Delta^{(1)}$ isometrically into $X^{(1)}$.

The following is part of the main theorem of [11].

**Theorem 7.5** ([11], Theorem 3.1)**.** *Let $X$ be a systolic complex. If $S$ is a wide flat minimal surface in $X$ then $S$ is almost geodesic.*

We will also use the following handy fact, whose proof can be extracted from [12]. In case $\gamma$ has length 2 it follows immediately from 6–largeness.

**Proposition 7.6** ([12], Proposition 3.10)**.** *Let $X$ be a systolic complex and $S\colon \Delta \to X$ a wide flat minimal surface. Let $\gamma$ be a neat 1–skeleton geodesic in $\Delta \subset \mathbb{E}^2_\Delta$, which is contained in a straight line. Then for any 1–skeleton geodesic $\overline{\gamma}$ in $X$ with the same endpoints as $S(\gamma)$ there is another minimal surface $S'\colon \Delta \to X$ such that $S'(\gamma) = \overline{\gamma}$ and $S = S'$ on the vertices of $\Delta$ outside $\gamma$.*

# 8 Layers

In this section we introduce and study the notion of layers for a pair of convex subcomplexes of a systolic complex. If those subcomplexes are vertices $v, w$, then the *layer* $k$ is the span of all vertices, in 1–skeleton geodesics $vw$, at distance $k$ from $v$ (c.f. Definition 8.1). In particular, simplices of the directed geodesics between $v$ and $w$ (c.f. Definition 2.11), as well as the simplices of Euclidean geodesics (which we construct in Section 9) lie in appropriate layers.

On the other hand, layers in systolic complexes seem to be interesting on their own.

**Definition 8.1.** Let $V, W$ be convex subcomplexes of a systolic complex $X$ and $n = |V, W|$. For $i = 0, 1, \ldots, n$ we define the *layer* $i$ between $V$ and $W$ as the subcomplex of $X$ equal to $B_i(V) \cap B_{n-i}(W)$. We will denote it by $L_i(V, W)$ (or shortly $L_i$ when $V, W$ are understood).

**Remark 8.2.** $L_i$ are convex, since they are intersections of convex $B_i(V), B_{n-i}(W)$ (see remarks after Definition 2.4).

**Lemma 8.3.**

*(i)* $L_i = S_i(V) \cap S_{n-i}(W)$, *for* $0 \le i \le n$.

*(ii)* $L_j \subset S_{j-i}(L_i)$, *for* $0 \le i < j \le n$. *In particular* $L_{i+1} \subset S_1(L_i)$, *for* $0 \le i < n$.

**Proof.** (i) W.l.o.g we only need to prove that $L_i \subset S_i(V)$. Take a vertex $x \in L_i$. Then we have $|x, V| \le i$ and $|x, W| \le n - i$, while $|V, W| = n$. Thus by the triangle inequality we have $|x, V| = i$, as desired.

(ii) By (i) we have that $B_{j-i-1}(L_i) \cap L_j = \emptyset$, thus we only need to prove that $L_j \subset B_{j-i}(L_i)$. Let $x$ be a vertex in $L_j$. Since, by (i), we have $x \in S_j(V)$, there is a vertex $y \in B_i(V)$ at distance $j - i$ from $x$. Since $x \in B_{n-j}(W)$, we have $y \in B_{n-i}(W)$. Thus $y \in L_i$ and $x \in B_{j-i}(L_i)$. □

Now we study the properties of layers.

**Lemma 8.4.** *For* $0 < i < n$ *we have that* $L_i$ *is* $\infty$*–large.*

**Proof.** Suppose the layer $L_i$ is not $\infty$–large. Then there exists an embedded cycle $\Gamma$ in $L_i$ (denote its consecutive vertices by $p_1, p_2, \ldots, p_k, p_1, k \ge 4$) which is a full subcomplex of $X$.

Denote $D_1 = \mathrm{span}\{B_{i-1}(V), \Gamma\}, D_2 = \mathrm{span}\{B_{n-i-1}(W), \Gamma\}$. We have that $D_1 \cap D_2 = \Gamma$. Notice that $D_1 \cup D_2$ is a full subcomplex of $X$, because there are no edges in $X$ between vertices in $B_{i-1}(V)$ and vertices in $B_{n-i-1}(W)$.

Observe that $\Gamma$ is contractible in $D_1$ (and similarly in $D_2$). Indeed, by Lemma 8.3(i) we have that $\Gamma \subset S_i(V)$. Thus we can project the edges of $\Gamma$ onto $B_{i-1}(V)$ (c.f. Definition 2.9). If we choose a vertex in each of these projections, we get, by Lemma 2.8, that these vertices form a loop. This loop is homotopic to $\Gamma$ in $D_1$. Moreover, since $B_{i-1}(V)$ is contractible (by remarks after Definition 2.4) it follows that $\Gamma$ is contractible in $D_1$ (and similarly in $D_2$), as desired. The simplicial sphere $S$ formed of these two contractions is contractible in $D_1 \cup D_2$ as full subcomplexes of $X$ are aspherical (Lemma 2.12).

Now use Meyer-Vietoris sequence of the pair $D_1, D_2$. Since $[\Gamma]$ is the image of $[S] = 0$ under $H_2(D_1 \cup D_2) \to H_1(D_1 \cap D_2)$ it follows that the cycle $\Gamma$ is homological to zero in itself. This is a contradiction. □

**Lemma 8.5.** *Let* $\sigma_1, \sigma_2, \sigma_3$ *be maximal simplices in the layer* $L_i$ *for some* $0 \le i \le n$ *and* $\tau_1 = \sigma_1 \cap \sigma_2,\ \tau_2 = \sigma_2 \cap \sigma_3$. *Then* $\tau_1 \cap \tau_2 = \emptyset$ *or* $\tau_1 \subset \tau_2$ *or* $\tau_2 \subset \tau_1$.

**Proof.** W.l.o.g. assume that $i \neq 0$. Suppose the lemma is false. Then there exist vertices $p_1 \in \tau_1 \backslash \tau_2,\ p_2 \in \tau_2 \backslash \tau_1,\ r \in \tau_1 \cap \tau_2$. By Lemma 8.3(ii) we have that $\sigma_1, \sigma_3 \subset S_1(L_{i-1})$. Denote by $q_1, q_2$ some vertices in the projections (c.f. Definition 2.9) of $\sigma_1, \sigma_3$ onto $L_{i-1}$. We have $|q_1 q_2| \leq 1$, because both $q_1$ and $q_2$ are neighbors of $r$ and the projection of $r \in L_i \subset S_1(L_{i-1})$ (c.f. Lemma 8.3(ii)) onto $L_{i-1}$ is a simplex (Lemma 2.8). Now we will argue that we can assume that $q_1 p_2$ is an edge. In case $q_1 \neq q_2$ consider the 4–cycle $q_1 q_2 p_2 p_1 q_1$. It must have a diagonal. We can then assume w.l.o.g. that $q_1 p_2$ is an edge. In case $q_1 = q_2$ we also have that $q_1 p_2$ is an edge. In both cases it follows that $p_2$ belongs to the simplex which is the projection of $q_1 \in L_{i-1} \subset S_1(L_i)$ (c.f. Lemma 8.3(ii)) onto $L_i$. This simplex also contains $\sigma_1$. But $p_2 \notin \sigma_1$, which contradicts the maximality of $\sigma_1$. □

**Corollary 8.6.** *Let $T$ be the following simplicial complex: the trapezoid build of three triangles $p_1 r s_1, p_1 r p_2, p_2 r s_2$. Then there is no isometric embedding of $T^{(1)}$ into $L_i^{(1)}$, for $0 \leq i \leq n$.*

**Proof.** Extend the images of those three triangles to maximal simplices $\sigma_1, \sigma_2, \sigma_3$ and apply Lemma 8.5. □

**Corollary 8.7.** *Let $0 < i < n$. Let $|p_0 r_0| \leq 1, |p_d r_d| \leq 1$ for vertices $p_0, r_0, p_d, r_d \in L_i$ such that $|p_0 p_d| = |r_0 r_d| = d \geq 2$ and $|p_0 r_d| \geq d, |r_0 p_d| \geq d$. Then, for any 1–skeleton geodesics $(p_i), (r_i), 0 \leq i \leq d$ connecting $p_0$ with $p_d$ and $r_0$ with $r_d$, respectively, and for any $0 \leq i, j \leq d$ such that $|i - j| \leq 1$, we have that $|p_i r_j| \leq 1$ (i.e. $p_i r_j$ is an edge or $p_i = r_j$).*

**Proof.** We will prove the corollary by induction on $d$. First observe that since $L_i$ is $\infty$–large (Lemma 8.4), the loop $p_0 p_1 \ldots p_d r_d \ldots r_1 r_0 p_0$ is triangulable and there exists a diagonal cutting off a triangle. There are only four possibilities for this diagonal and we can w.l.o.g suppose this diagonal is $p_0 r_1$. Now since $p_0 \in S_d(r_d p_d)$ and both $p_1$ and $r_1$ lie in the projection of $p_0$ onto $B_{d-1}(r_d p_d)$, then by Lemma 2.8 either $p_1 r_1$ is an edge or $p_1 = r_1$.

Now we start the induction. If $d = 2$ and the loop $p_1 r_1 r_2 p_2 p_1$ is embedded, then it has a diagonal. The rest of the required inequalities follows from applying twice Corollary 8.6.

Suppose that for $d - 1$ the corollary is already proved. Then applying it to the pair $p_1 r_1, p_d r_d$ yields all the required inequalities except for the estimate on $|r_0 p_1|$. But this follows from Corollary 8.6 applied to the trapezoid $r_0 p_0 r_1 p_1 p_2$. □

**Corollary 8.8.** *If $pr, p'r'$ are edges in $L_i$, for some $0 < i < n$, such that $|pp'| = |rr'| = d \geq 2$ and $|pr'| \leq d, |p'r| \leq d$, then $|pr'| = |p'r| = d$.*

**Proof.** By contradiction.

**Case** $|pr'| = |p'r| = d - 1$**.** If $d > 2$ (if $d = 2$ there is a diagonal in the square $pr'p'rp$) then Corollary 8.7 applied to $d-1$ in place of $d$, $p_0 = p$ , $p_{d-1} = r'$, $r_0 = r$, $r_{d-1} = p'$ gives $|pp'| = |rr'| = d-1$, contradiction.

**Case** $|pr'| = d - 1$, $|p'r| = d$**.** Again apply Corollary 8.7, with $p_0 = p$, $r_0 = r$, $p_d = r_d = p'$, $p_{d-1} = r'$, getting $|rr'| = d - 1$, contradiction. □

Below we present another important property of layers. Since it will not be needed in the article, we do not include the proof. Denote $L = \mathrm{span}(L_i \cup L_{i+1})$ for some $1 \leq i < n-1$.

**Lemma 8.9.** *$L$ is $\infty$–large.*

We end with a simple, but useful observation.

**Lemma 8.10.** *For any edges $vw, xy$ such that $v, x \in L_i$, $w, y \in L_{i+1}$, where $0 \leq i < n$, we have that $||vx| - |wy|| \leq 1$.*

**Proof.** By contradiction. Suppose, w.l.o.g., that $|wy| = 2 + |vx|$. Hence $v$ lies on a 1–skeleton geodesic $wy$. Thus, by convexity of layers (Remark 8.2) and by Proposition 2.6, we have that $v$ lies in $L_{i+1}$, which is, by Lemma 8.3, disjoint with $L_i$, contradiction. □

# 9 Euclidean geodesics

In this section we define, for a pair of simplices $\sigma, \tau$ as below, a sequence of simplices in the layers between $\sigma$ and $\tau$, which can be considered as a "Euclidean" geodesic joining $\sigma$ and $\tau$. Unlike the directed geodesics defined by Januszkiewicz and Świątkowski (see Definition 2.11), Euclidean geodesics are symmetric with respect to $\sigma$ and $\tau$.

The definition requires a lenghty preparation. Roughly speaking, we start by spanning a minimal surface between directed geodesics from $\sigma$ to $\tau$ and from $\tau$ to $\sigma$. We observe that this surface is flat whenever the two directed geodesics are far apart (we call the corresponding layers *thick*). Next we show that this "piecewise" flat surface is in some sense unique. This occupies the first part of the section, up to Definition 9.9. Then we look at the geodesics in the Euclidean metric in the flat pieces and use them to define *Euclidean geodesics* in systolic complexes, c.f. Definition 9.12. Finally, we establish some of their basic properties.

The setting, which we fix for Sections 9—13 is the following. Let $\sigma, \tau$ be simplices of a systolic complex $X$, such that for some natural $n \geq 0$ we have $\sigma \subset S_n(\tau), \tau \subset S_n(\sigma)$. Let $\sigma_0 \subset \sigma, \sigma_1, \ldots, \sigma_n \subset \tau$ and $\tau_n \subset \tau, \tau_{n-1}, \ldots, \tau_0 \subset$

$\sigma$ be sequences of simplices in $X$, such that for $0 \leq k < n$ we have that $\sigma_k, \sigma_{k+1}$ span a simplex and $\tau_k, \tau_{k+1}$ span a simplex. In particular, $\sigma_k, \tau_k$ lie in the layer $k$ between $\sigma$ and $\tau$ (c.f. Definition 8.1).

Note that if $\sigma_0 = \sigma, \sigma_1, \ldots, \sigma_n \subset \tau$ is the directed geodesic from $\sigma$ to $\tau$ and $\tau_n = \tau, \tau_{n-1}, \ldots, \tau_0 \subset \sigma$ is the directed geodesic from $\tau$ to $\sigma$ (c.f. Definition 2.11), then the above condition is satisfied. This special choice of $(\sigma_k), (\tau_k)$ will be very important later and we will frequently distinguish it.

**Definition 9.1.** For $0 \leq i \leq n$ the *thickness* of the layer $i$ for $(\sigma_k), (\tau_k)$ is the maximal distance between vertices in $\sigma_i$ and in $\tau_i$. If the layer $i$ for $(\sigma_k), (\tau_k)$ has thickness $\leq 1$ we say that the layer $i$ for $(\sigma_k), (\tau_k)$ is *thin*, otherwise we say that the layer $i$ for $(\sigma_k), (\tau_k)$ is *thick*. If $(\sigma_k), (\tau_k)$ are directed geodesics from $\sigma$ to $\tau$ and from $\tau$ to $\sigma$, respectively, then we skip "for $(\sigma_k), (\tau_k)$" for simplicity.

**Caution.** Perhaps, to avoid confusion, we should not have used the word "layer" in the above definition, since we are in fact only checking the position of $\sigma_i$ w.r.t. $\tau_i$. Even if the layer $i$ between $\sigma$ and $\tau$ is large, it can happen that the thickness of the layer $i$ for $(\sigma_k), (\tau_k)$ is small. However, we decided that this terminology suits well our approach, in which we will be mostly interested in the part of the layer $i$ between $\sigma$ and $\tau$, which lies between $\sigma_i$ and $\tau_i$.

**Definition 9.2.** A pair $(i, j)$, where $0 \leq i < j \leq n$ is called a *thick interval* (for $(\sigma_k), (\tau_k)$) if the layers $i$ and $j$ (for $(\sigma_k), (\tau_k)$) are thin, $i + 1 < j$, and for every $l$, such that $i < l < j$, the layer $l$ (for $(\sigma_k), (\tau_k)$) is thick. We say that the thick interval $(i, j)$ *contains* $l$ if $i < l < j$.

**Lemma 9.3.**

*(i) The thickness of consecutive layers varies at most by 1.*

*(ii) If $(i, j)$ is a thick interval (for $(\sigma_k), (\tau_k)$), then $\sigma_i, \tau_i$ are disjoint.*

**Proof.** Both parts follow immediately from Lemma 8.10. □

**Definition 9.4.** Let $(i, j)$ be a thick interval (for $(\sigma_k), (\tau_k)$). Let vertices $s_k \in \sigma_k$, $t_k \in \tau_k$ be such that for each $i \leq k \leq j$ the distance $|s_k t_k|$ is maximal (i.e. $s_k, t_k$ realize the thickness of the layer $k$). By Lemma 9.3(ii) the sequence $s_i, s_{i+1}, \ldots, s_j, t_j, t_{j-1}, \ldots t_i, s_i$ is an embedded loop, thus we can consider a minimal surface $S \colon \Delta \to X$ spanned on this loop (c.f. Definition 7.4). We say that $S$ is a *characteristic surface* (for the thick interval $(i, j)$) and $\Delta$ is a *characteristic disc.*

**Lemma 9.5.** *For $s_k, s'_k \in \sigma_k,\ t_k, t'_k \in \tau_k$, if distances $|s_k t'_k|, |s'_k t_k|$ equal the thickness of the layer $k$ then also $|s_k t_k|$ equals the thickness of the layer $k$, i.e. if vertices $s_k \in \sigma_k, t_k \in \tau_k$ realize the thickness in some pairs, then they also realize the thickness as a pair.*

**Proof.** Immediate from definition of thickness and Corollary 8.8. □

The lemma below summarizes the geometry of characteristic discs, which we need to introduce the concept of a Euclidean geodesic. The special features of characteristic discs, in the case that $(\sigma_k), (\tau_k)$ are directed geodesics, will be given in Lemma 9.16 at the end of this section.

Let $S\colon \Delta \to X$ be a characteristic surface. Denote by $v_k, w_k \in \Delta$ the preimages of $s_k, t_k$ in $X$, respectively. This notation will be fixed for the entire article. Let us point out that we use numbers $i, \ldots, j$ to number the layers in $\Delta$ (c.f. Definition 8.1) between $v_i w_i$ and $v_j w_j$, instead of $0, \ldots j - i$, for the sake of clarity.

**Lemma 9.6.**

*(i) $\Delta$ (and thus the characteristic surface $S$) is wide and flat,*

*(ii) if we embed $\Delta \subset \mathbb{E}^2_\Delta$, then the edges $v_i w_i$ and $v_j w_j$ are parallel and consecutive layers between them are contained in consecutive straight lines (treated as subcomplexes of $\mathbb{E}^2_\Delta$) parallel to the lines containing $v_i w_i$ and $v_j w_j$.*

**Proof.** (i) To prove wideness it is enough to show that any nonconsecutive vertices of the boundary loop are at distance $\geq 2$. Since the layers $k$, where $i < k < j$, are thick (for $(\sigma_k), (\tau_k)$), the only possibility for this to fail is that (w.l.o.g.) $|s_k t_{k+1}| = 1$ for some $i < k < j$. If this happens, then both $s_k$ and $t_k$ lie in the projection of $t_{k+1}$ onto the layer $k$ between $\sigma$ and $\tau$ (the projection is defined by Lemma 8.3(ii)), hence they are neighbors (Lemma 2.8), which contradicts $|s_k t_k| \geq 2$. Thus a characteristic disc is wide.

Before proving flatness, we need the following general observation. If $\Gamma$ is a 1–skeleton geodesic, which is in the boundary of a triangulation of a disc, then the sum of the defects at the vertices in the interior of $\Gamma$ is $\leq 1$. Moreover, all the defects at these vertices are $\leq 1$ and each two vertices of positive defect are separated by a vertex of negative defect.

To prove flatness, compute possible defects at the boundary vertices of $\Delta$. By wideness, they are $\leq 1$ at $v_i, v_j, w_i, w_j$. Moreover, their sum over the interior vertices of each of the 1–skeleton geodesics $(v_k)_{k=i}^j, (w_k)_{k=i}^j$ is $\leq 1$ (they are 1–skeleton geodesics, since their images are). Thus Gauss–Bonnet Lemma 7.2 implies that the defects of the interior vertices are equal to zero, the sums of the defects over the vertices $(v_k)_{k=i+1}^{j-1}$, $(w_k)_{k=i+1}^{j-1}$ equal 1 each and the defects at $v_i, v_j, w_i, w_j$ are equal to 1.

We now want to say more about the defects at $(v_k)_{k=i+1}^{j-1}$. Up to now we know that their sum is 1, they equal $1, 0, -1$ or $-2$ and each two vertices of positive defect are separated by a vertex of negative defect (since $(v_k)_{k=i}^{j}$ is a 1–skeleton geodesic). This implies that the defects equal alternatingly $1, -1, 1-1, \ldots, 1$ with possible 0's between them. The same holds for the defects at $(w_k)_{k=i+1}^{j-1}$. Thus, by Lemma 7.3 (characterization of flatness), the characteristic disc $\Delta$ is flat, i.e. we have an embedding $\Delta \subset \mathbb{E}^2_\Delta$ isometric on the 1–skeleton.

(ii) By the computation of defects in the proof of (i) we get that the edges $v_iw_i$ and $v_jw_j$ are parallel in $\mathbb{E}^2_\Delta$. We also get that $v_k, w_k$, for $i \leq k \leq j$ are at combinatorial distances $k-i, j-k$ from the lines containing the edges $v_iw_i, v_jw_j$. Hence $v_k, w_k$ lie on the appropriate line parallel to $v_iw_i$ and the vertices of $\Delta$ split into families lying on geodesics $v_kw_k$. By convexity of layers, Remark 8.2, (or by direct observation) these geodesics are equal to the layers. □

When speaking about the layers in $\Delta$ between $v_iw_i$ and $v_jw_j$, we will often skip "between $v_iw_i$ and $v_jw_j$".

**Remark 9.7.** Denote the layer $k$ in $\Delta$ (between $v_iw_i$ and $v_jw_j$) by $L_k$. Then $S(L_k)$ is contained in the layer $k$ in $X$ between $\sigma$ and $\tau$. This follows from

$$\begin{aligned} S(L_k) &\subset S(B_{k-i}(v_iw_i)) \cap S(B_{j-k}(v_jw_j)) \subset \\ &\subset B_{k-i}(S(v_iw_i)) \cap B_{j-k}(S(v_jw_j)) \subset \\ &\subset B_{k-i}(\sigma_i\tau_i) \cap B_{j-k}(\sigma_j\tau_j) \subset \\ &\subset B_{k-i}(B_i(\sigma)) \cap B_{j-k}(B_{n-j}(\tau)) = B_k(\sigma) \cap B_{n-k}(\tau). \end{aligned}$$

The next lemma summarizes some uniqueness properties of characteristic surfaces for a fixed thick interval $(i, j)$.

**Lemma 9.8.**

*(i) A characteristic surface is almost geodesic. In particular, it is an isometric embedding on the 1–skeleton of a subcomplex spanned by any pair of consecutive layers between $v_iw_i$ and $v_jw_j$ in $\Delta$.*

*(ii) A characteristic disc $\Delta \subset \mathbb{E}^2_\Delta$ does not depend (up to isometry) on the choice of $s_k, t_k$ and the choice of a characteristic surface.*

*If we have two characteristic surfaces $S_1 \colon \Delta_1 \to X, S_2 \colon \Delta_2 \to X$, then after identifying the characteristic discs $\Delta_1 = \Delta_2$ (which is possible by (ii)) we have that* *(iii) for any vertices $x, y \in \Delta_1 = \Delta_2$ at distance 1, $S_1(x)$ and $S_2(y)$ are also at distance 1, i.e. for any two characteristic surfaces $S_1, S_2$ we can substitute an image of a vertex of the first surface with the corresponding image in the second and get another characteristic surface,*

*(iv) for any vertex $x \in \Delta_1 = \Delta_2$, $S_1(x)$ and $S_2(x)$ are at distance $\leq 1$.*

**Proof.** (i) This follows from Elsner's Theorem 7.5, since, by Lemma 9.6(i), a characteristic disc is flat and wide. The second part follows from the fact that any two vertices in a same or consecutive layers in $\Delta \subset \mathbb{E}^2_\Delta$ can be connected by a neat geodesic, which can be verified by direct observation.

(ii) Observe that, by Lemma 9.6(ii), the isometry class of $\Delta$ is determined by the distances $|v_k w_k|$, $|v_k w_{k+1}|$, for $i \leq k \leq j-1$, which are equal, by (i), to $|s_k t_k|, |s_k t_{k+1}|$, respectively. The value $|s_k t_k|$ equals the thickness of the layer $k$, so it does not depend on the choices. To prove the same for $|s_k t_{k+1}|$, consider two characteristic surfaces constructed for choices $s_l, s'_l \in \sigma_l,\ t_l, t'_l \in \tau_l$, where $l = k, k+1$. We will prove that $|s_k t_{k+1}| = |s_k t'_{k+1}| = |s'_k t'_{k+1}|$. We restrict ourselves to proving the first equality (the second is proved analogically). By Lemma 9.5 we have that $|s_{k+1} t'_{k+1}|$ is the thickness of the layer $k+1$. Thus there is a characteristic surface spanned on a loop passing through $s_k, t_k, s_{k+1}, t'_{k+1}$. Hence, by (i), the distance $|s_k t'_{k+1}|$ is determined by $|s_k t_k|$ and $|s_{k+1} t_k|$, thus it is the same as $|s_k t_{k+1}|$, as desired.

(iii) If $x$ and $y$ are both boundary vertices, then this is obvious. Otherwise, w.l.o.g. assume that $x$ is an interior vertex of $\Delta$. Suppose $x$ lies in the layer $k$ (we denote it by $L_k$) in $\Delta$ between $v_i w_i$ and $v_j w_j$. Denote the thickness of the layer $k$ for $(\sigma_t), (\tau_t)$ by $d$.

First consider the case that $y \in L_k$. By Remark 9.7 we have that $S_1(L_k)$ and $S_2(L_k)$ lie in the layer $k$ in $X$ between $\sigma$ and $\tau$. By Lemma 9.5 we have that $|S_2(v_k)S_1(w_k)| = |S_1(v_k)S_2(w_k)| = d$. Hence Corollary 8.7 applied to $S_1(L_k)$ and $S_2(L_k)$ gives $|S_1(x)S_2(y)| = 1$, as desired.

Now, w.l.o.g., consider the remaining case that $y$ is in the layer $k-1$ (denoted by $L_{k-1}$) in $\Delta$ between $v_i w_i$ and $v_j w_j$. Denote by $y', x''$ the common neighbors of $x, y$ in $L_{k-1}, L_k$, respectively, and by $x'$ the neighbor of $x$ in $L_k$ different from $x''$. Then, from the previous case, we have that $S_1(x)S_2(x')S_2(y')S_2(y)S_2(x'')S_1(x)$ is a loop of length 5, hence it is triangulable. By (i), all $|S_2(x')S_2(x'')|$, $|S_2(x')S_2(y)|$, $|S_2(x'')S_2(y')|$ equal 2, hence we obtain $|S_1(x)S_2(y)| = 1$, as desired.

Observe that this proof actually implies Proposition 7.6 in the case that $\gamma \subset v_k w_k$ for some $k$.

(iv) For boundary vertices this is obvious. For an interior vertex $x$, let $x', x''$ be its neighbors in a common layer in $\Delta$ between $v_i w_i, v_j w_j$. Then, by (iii), we have that $S_1(x)S_2(x')S_2(y)S_2(x'')S_1(x)$ is a loop of length 4. Moreover, by (i), we have that $|S_2(x')S_2(x'')| = 2$. Thus $|S_1(x)S_2(y)| \leq 1$, as desired. $\square$

As a corollary, the following definition is allowed.

**Definition 9.9.** Let $\rho$ be a simplex of the characteristic disc $\Delta$ for some thick interval $(i, j)$ (for $(\sigma_k), (\tau_k)$). Its *characteristic image* is a simplex in $X$, denoted by $\mathcal{S}(\rho)$, which is the span of the images of $\rho$ under all possible characteristic surfaces. Note that $\mathcal{S}(\rho)$ is a simplex by Lemma 9.8(iii,iv), and if $\rho \subset \rho'$, then $\mathcal{S}(\rho) \subset \mathcal{S}(\rho')$, i.e. $\mathcal{S}$ respects inlucions. The *characteristic image* of a subcomplex of $\Delta$ is the union of the characteristic images of all its simplices. We call this assignment the *characteristic mapping*.

If $\overline{v}$ is a vertex in $\mathcal{S}(\Delta)$, we denote by $\mathcal{S}^{-1}(\overline{v})$ the vertex $v \in \Delta$ such that $\mathcal{S}(v) \ni \overline{v}$. We claim that this vertex is unique. Indeed, characteristic images of different layers in $\Delta$ between $v_iw_i, v_jw_j$ are disjoint since, by Remark 9.7, they lie in different layers in $X$ between $\sigma, \tau$, disjoint by Lemma 8.3. Moreover, by Lemma 9.8(i,iii), we have that $S_1(v) \neq S_2(v')$ for any characteristic surfaces $S_1, S_2$ and any vertices $v \neq v'$ in a common layer in $\Delta$. This justifies the claim. If $\overline{\rho}$ is a simplex in $\mathcal{S}(\Delta)$, we denote by $\mathcal{S}^{-1}(\overline{\rho})$ the span of the union of $\mathcal{S}^{-1}(\overline{v})$ over all $\overline{v} \in \overline{\rho}$. We have that $\mathcal{S}^{-1}(\overline{\rho})$ is a simplex, by Remark 9.7, Lemma 8.3, and Lemma 9.8(i,iii). If $Y$ is a subcomplex of $\mathcal{S}(\Delta)$, we denote by $\mathcal{S}^{-1}(Y)$ the union of $\mathcal{S}^{-1}(\overline{\rho})$ over all $\overline{\rho} \subset Y$.

Having established the uniqueness properties of characteristic surfaces, we start to exploit the $CAT(0)$ structure of the corresponding characteristic discs. From now on, up to the end of Section 13, unless stated otherwise, assume that $(\sigma_k), (\tau_k)$ are the directed geodesics between $\sigma, \tau$.

**Definition 9.10.** Let $(i, j)$ be a thick interval and let $\Delta \subset \mathbb{E}^2_\Delta$ be its characteristic disc. We will define a sequence of simplices $\rho_k \in \Delta$, where $i < k < j$, which will be called the *Euclidean diagonal* of the characteristic disc $\Delta$.

Let $v'_k, w'_k$ be points (barycenters of edges) on the straight line segments $v_kw_k$ at distance $\frac{1}{2}$ from $v_k, w_k$, respectively. In particular $v'_i = w'_i$, $v'_j = w'_j$. Consider the closed polygonal domain $\Delta' \subset \Delta$ enclosed by the piecewise linear loop with consecutive vertices $v'_i, v'_{i+1}, \ldots, v'_j = w'_j, w'_{j-1}, \ldots, w'_i = v'_i$. Note that, since $\Delta'$ is simply–connected, it is $CAT(0)$ with the Euclidean path metric induced from $\mathbb{E}^2_\Delta$ identified with $\mathbb{E}^2$. We call $\Delta'$ a *modified characteristic disc*. Let $\gamma'$ be the $CAT(0)$ geodesic joining $v'_i = w'_i$ to $v'_j = w'_j$ in $\Delta'$. We call $\gamma'$ a *CAT(0) diagonal* of $\Delta$. For each $i < k < j$, among the vertices of $\Delta$ lying in the interior of the 1–skeleton geodesic $v_kw_k$ find the ones nearest to $\gamma' \cap v_kw_k$. For each $k$ this is either a single vertex or two vertices spanning an edge (if $\gamma'$ goes through its barycenter and $v_k, w_k$

are not some of its vertices). We put $\rho_k$ equal to this vertex or this edge, accordingly.

At first sight it might seem strange that in the above definition we pass to $\Delta'$ and take the geodesic $\gamma'$ there instead of doing it in $\Delta$ itself. However, this construction allows us to exclude $v_k, w_k$ from being in $\rho_k$, which a careful reader will find to be a necessary condition for the arguments of the combinatorial Proposition 10.2 to be valid.

Here are some basic properties of the Euclidean diagonals.

**Lemma 9.11.**

*(i) Each pair of consecutive $\rho_k, \rho_{k+1}$, for $i < k < j-1$, spans a simplex.*

*(ii) $\rho_{i+1}, v_i, w_i$ span a simplex and $\rho_{j-1}, v_j, w_j$ span a simplex.*

**Proof.** Part (ii) is obvious, since we excluded $v_k, w_k$ from being in $\rho_k$. To prove (i), consider $\Delta' \subset \Delta \subset \mathbb{E}^2_\Delta$ oriented in such a way that $v_k w_k$ are horizontal, this is possible by Lemma 9.6(ii). Moreover, Lemma 9.6(ii) yields that the boundary of $\Delta'$ consists of line segments at angle 30° from the vertical direction. Let $\gamma'$ be as in Definition 9.10. It is a broken line with vertices at the boundary of $\Delta'$.

We claim that any line segment of $\gamma'$ is at angle $< 30°$ from the vertical direction. First we prove that this angle is $\leq 30°$. Otherwise, let $p$ be an endpoint of such a line segment. Obviously $p$ is different from the endpoints of $\gamma'$. The interior angle at $p$ between the segment of $\gamma'$ and any of the boundary line segments of $\Delta'$ is $< 180°$, which contradicts the fact that $p$ is an interior vertex of a geodesic $\gamma'$. Thus we proved that any line segment of $\gamma'$ is at angle $\leq 30°$ from the vertical direction.

If for some line segment of $\gamma'$ this angle equals 30°, then by the previous considerations the whole $\gamma'$ is in fact a straight line at angle 30° from the vertical. This implies that the defects at all vertices in $(v_k)_{k=i+1}^{j-1}$ or all vertices in $(w_k)_{k=i+1}^{j-1}$ are zero. Contradiction.

Now part (i) follows from the following observation, whose proof is easy and is left for the reader. Consider two consecutive horizontal lines $\alpha_1, \alpha_2$ in $\mathbb{E}^2_\Delta$. Let $\beta$ be some straight line segment joining points $p \in \alpha_1, r \in \alpha_2$ at angle $< 30°$ from the vertical direction. Then there exist two 2–simplices $abc, bcd$ in $\mathbb{E}^2_\Delta$ such that $ab \subset \alpha_1, cd \subset \alpha_2$ and $p \in ab, r \in cd$. Moreover, it cannot happen simultaneously that $|pa| \leq |pb|$ and $|rd| \leq |rc|$. □

Thus we can finally introduce the main definition of this section.

**Definition 9.12.** We define a sequence of simplices $\delta_k$, where $0 \leq k \leq n$, which is called the *Euclidean geodesic* between $\sigma, \tau$, as follows. For each $k$, if the layer $k$ is thin, then we take $\delta_k$ as the span of $\sigma_k$ and $\tau_k$.

If the layer $k$ is thick, consider the thick interval $(i, j)$ which contains $k$. Let $\rho_k$ be an appropriate simplex of the Euclidean diagonal of the characteristic disc $\Delta$ for $(i, j)$ (c.f. Definition 9.10). We take $\delta_k = \mathcal{S}(\rho_k)$ (c.f. Definition 9.9).

**Remark 9.13.** In the above setting, we have $\sigma_i = \mathcal{S}(v_i)$, $\tau_i = \mathcal{S}(w_i)$, by Lemma 9.3(ii). Hence $\delta_i = \text{span}\{\sigma_i, \tau_i\} = \mathcal{S}(v_i w_i)$.

**Remark 9.14.** By the symmetry of the construction, the Euclidean geodesic between $\sigma$ and $\tau$ becomes the Euclidean geodesic between $\tau$ and $\sigma$ if we take the simplices of the sequence in the opposite order.

Here is the justification for using the name "geodesic" in Definition 9.12.

**Lemma 9.15.**

*(i) For any* $0 \leq k < l \leq n$ *we have that* $\delta_k \subset S_{l-k}(\delta_l), \delta_l \subset S_{l-k}(\delta_k)$.

*(ii) For any* $0 \leq k \leq n-1$ *if the layer* $k$ *or the layer* $k+1$ *is thick, then* $\delta_k$ *and* $\delta_{k+1}$ *span a simplex.*

*(iii) For any* $0 \leq l < m \leq n$ *such that there exists* $l \leq k \leq m$ *such that the layer* $k$ *is thick, and for any vertices* $x \in \delta_m, y \in \delta_l$, *we have* $|xy| = m - l$.

**Proof.** Assertion (ii) follows from Lemma 9.11(i,ii), Remark 9.13 and Lemma 9.8(iii,iv).

To prove assertion (i), say the first inclusion, observe that for any $0 \leq k < n$ we have $\text{span}(\sigma_k \cup \tau_k) \subset B_1(\text{span}(\sigma_{k+1} \cup \tau_{k+1}))$. Hence, assertion (ii) gives already, for any $0 \leq k < l \leq n$, that $\delta_k \subset B_{l-k}(\delta_l)$. Then $\delta_k \subset S_{l-k}(\delta_l)$ follows from Remark 9.7 and Lemma 8.3(ii).

To prove part (iii), assume that $l < k < m$ (other cases are easier). Take any vertex $z \in \delta_k$. Then, by (i), there are vertices $x' \in \delta_{k-1},\ y' \in \delta_{k+1}$ such that $|xx'| = (k-1) - l,\ |yy'| = m - (k+1)$. By (ii) (and (i)), we have $|zx'| = |zy'| = 1$. Hence $|xy| \leq m - l$ and by (i) we have $|xy| = m - l$, as desired. □

Now we state an extra property of characteristic discs in the case that $(\sigma_k)$ (but $(\tau_k)$ not necessarily) is the directed geodesic. This property was not necessary for Definition 9.12, but will become indispensable in the next section.

**Lemma 9.16.**

*(i) If the defect at some $v_k$, where $i+1 < k < j-1$, equals $-1$, then the defect at $v_{k+1}$ equals 1.*

*(ii) The defect at $v_{i+1}$ equals 1.*

**Proof.** (i) Proof by contradiction. Suppose the defect at some $v_k$, where $i+1 < k < j-1$, equals $-1$, and the defect at $v_{k+1}$ equals 0. Denote by $x$ the vertex next to $v_{k+1}$ on the 1–skeleton geodesic $v_{k+1}w_{k+1}$ and by $y$ the vertex next to $v_k$ on the 1–skeleton geodesic $v_k w_k$. We aim to prove that, for any characteristic surface $S$, $S(x)$ belongs to $\sigma_{k+1}$. Suppose for a moment we have already proved this. Then, since by Lemma 9.8(i) we have $|S(x)S(v_{k+2})| = 2$ and at the same time $S(v_{k+2}) \in \sigma_{k+2}$, we get a contradiction.

Now we prove that $S(x) \in \sigma_{k+1}$. By Remark 9.7, $S(x)$ lies in the layer $k+1$ between $\sigma$ and $\tau$. Now by definition of projection (c.f. Definition 2.9) we need to prove that $S(x)$ is a neighbor of each $\bar{z} \in \sigma_k$. Case $\bar{z} = S(y)$ is obvious, so suppose $\bar{z} \neq S(y)$. Since, by definition of thickness, $|\bar{z}S(w_k)| \leq |S(v_k)S(w_k)|$, we have by Lemma 8.7 (applied to $r_0 = S(v_k), r_1 = S(y), r_d = p_d = S(w_k)$ and to $p_0 = \bar{z}$ in case of $|\bar{z}S(w_k)| = |S(v_k)S(w_k)|$ or to $p_0 = S(v_k), p_1 = \bar{z}$ in case of $|\bar{z}S(w_k)| < |S(v_k)S(w_k)|$) that $|\bar{z}S(y)| = 1$. Considering the loop $\bar{z}S(y)S(x)S(v_{k+1})\bar{z}$, since $|S(y)S(v_{k+1})| = |yv_{k+1}| = 2$ (Lemma 9.8(i)), we get $|\bar{z}S(x)| = 1$, as desired.

(ii) By contradiction. Denote by $x$ the vertex between $v_{i+1}$ and $w_{i+1}$ on the 1–skeleton geodesic $v_{i+1}w_{i+1}$. Since $\sigma_i = \mathcal{S}(v_i)$ (see Remark 9.13), we have by Remark 9.7 and Lemma 9.8(iii) that $S(x)$ belongs to $\sigma_{i+1}$. By Lemma 9.8(i) we have $|S(x)S(v_{i+2})| = 2$. At the same time $S(v_{i+2}) \in \sigma_{i+2}$, contradiction. □

We will repeat some steps of this proof later on in the proof of Lemma 10.3. We decided, for clarity, not to interwind these two proofs.

As a consequence of Lemma 9.16, we get the following lemma, whose proof, similar to the proof of Lemma 9.11, we omit. Here we assume that both $(\sigma_k), (\tau_k)$ are directed geodesics.

**Lemma 9.17.** *If $j - i > 2$ then the $CAT(0)$ diagonal $\gamma'$ in $\Delta$ crosses each line orthogonal to the layers transversally.*

# 10 Directed geodesics between simplices of Euclidean geodesics

In this section we start to prove a weak version of Theorem B, which concerns one of the main properties of Euclidean geodesics. Roughly speaking, the theorem says that pieces of Euclidean geodesics are coarsely also Euclidean geodesics.

We keep the notation from the previous section. The simplices $(\sigma_k), (\tau_k)$ are in this section the directed geodesics between $\sigma, \tau$.

**Theorem 10.1** (weak version of Theorem B)**.** *Let $\sigma, \tau$ be simplices of a systolic complex $X$, such that for some natural $n$ we have $\sigma \subset S_n(\tau), \tau \subset S_n(\sigma)$ (as required in the definition of the Euclidean geodesic). Let $(\delta_k)_{k=0}^n$ be the Euclidean geodesic between $\sigma$ and $\tau$. Take some $0 \leq l < m \leq n$ and consider the simplices $\tilde{\delta}_l = \delta_l, \tilde{\delta}_{l+1}, \ldots, \tilde{\delta}_m = \delta_m$ of the Euclidean geodesic between $\delta_l$ and $\delta_m$ (we can define it by Lemma 9.15(i)). Then for each $l \leq k \leq m$ we have $|\delta_k, \tilde{\delta}_k| \leq 3$.*

The proof of Theorem 10.1 splits into two steps. The first step is to prove that directed geodesics between $\delta_l$ and $\delta_m$ stay close to the union of characteristic images of all characteristic discs (for $(\sigma_k), (\tau_k)$). This is the content of Proposition 10.2, whose proof occupies the rest of this section.

The second step is to check that characteristic images for the directed geodesics between $\delta_l$ and $\delta_m$ also stay close to the union of characteristic images for $(\sigma_k), (\tau_k)$. Properties of layers actually imply that characteristic discs of the former are embedded into characteristic discs of the latter, modulo small neighborhood of the boundary. So everything boils down to the fact that Theorem 10.1 is valid for $CAT(0)$ subspaces of the Euclidean plane. We carry out this program in the next section. We also indicate there an argument, how to promote Theorem 10.1 to Theorem B, with a reasonable constant $C$.

A complete alternative proof of Theorem B, with a worse constant $C$, is obtained as a consequence of Proposition 12.1. We present it at the end of Section 12. We advise the reader to have a look at the proof of Theorem 10.1 via Proposition 10.2. This proof is straightforward and allows us to introduce gradually some concepts needed later. However, to save time, one can skip the remaining part of Section 10, go over the definitions in Section 11 and then go directly to Section 12.

For each thick layer $l \leq k \leq m$ contained in a thick interval $(i, j)$ (for $(\sigma_t), (\tau_t)$; from now on we often skip "for $(\sigma_t), (\tau_t)$"), denote by $\alpha_k$ the appropriate simplex (in the corresponding characteristic disc $\Delta$) of the directed geodesic from $\rho_l$, if $i < l$, or $v_i$ otherwise, to $\rho_m$, if $m < j$, or $v_j$ otherwise. The simplices $(\tilde{\sigma}_k)_{k=l}^m$ of the directed geodesic from $\delta_l$ to $\delta_m$ satisfy the following.

**Proposition 10.2.** *Let $l \leq k \leq m$.*

*(i) If the layer $k$ is thin, then $\sigma_k$ contains or is contained in $\tilde{\sigma}_k$,*

*(ii) if the layer $k$ is thick, then $\mathcal{S}(\alpha_k)$ contains or is contained in $\tilde{\sigma}_k$.*

Before we give the proof of Proposition 10.2, we need to establish some necessary lemmas. The first one describes the position of $\sigma_k$ with respect to the characteristic image. Like in Lemma 9.16, here $(\tau_k)$ does not need to be the directed geodesic.

**Lemma 10.3.** *For a thick layer $k$ let $x_k$ be the vertex, which is a neighbor of $v_k$ on the 1–skeleton geodesic $v_k w_k$ in the characteristic disc for the thick interval containing $k$. If the defect at $v_k$ equals 1, then $\sigma_k = \mathcal{S}(v_k x_k)$. Otherwise $\sigma_k = \mathcal{S}(v_k)$.*

**Proof.** First of all $\sigma_k \subset \mathcal{S}(v_k x_k)$ follows from the definition of thickness and Proposition 7.6 (one could also verify this by hand, similarly like in the proofs of Lemma 9.6(iii) and Lemma 9.16(i)). Suppose the defect at $v_k$ is $\neq 1$. Hence $|v_{k-1} x_k| = 2$, by Lemma 9.16(i,ii). The inclusion $\mathcal{S}(v_k) \subset \sigma_k$ is obvious and the converse inclusion follows from $\sigma_k \subset \mathcal{S}(v_k x_k)$ and from Lemma 9.8(i).

Now suppose the defect at $v_k$ equals 1. If the layer $k-1$ is thick, then the defect at $v_{k-1}$ is $\neq 1$ and we apply what we have just proved to get $\mathcal{S}(v_{k-1}) = \sigma_{k-1}$. If the layer $k-1$ is thin we get immediately that $\mathcal{S}(v_{k-1}) = \sigma_{k-1}$ (Remark 9.13). In both cases using Remark 9.7, Lemma 9.8(iii), and the definition of projection we get $\mathcal{S}(v_k x_k) \subset \sigma_k$, as desired. □

As a corollary we get the following technical lemma.

**Lemma 10.4.** *Suppose $k < m$ do not satisfy $i \leq k < m < j$ for any thick interval $(i, j)$ or if they violate this then $|v_{k+1}, \rho_m| = m - (k+1)$. Then the projection of $\sigma_k$ onto $B_{m-(k+1)}(\delta_m)$ equals $\sigma_{k+1}$.*

**Proof.** To justify speaking about the projection of $\sigma_k$ onto $B_{m-(k+1)}(\delta_m)$ we must show that $\sigma_k \subset S_{m-k}(\delta_m)$. The simplex $\sigma_k$ is outside $B_{m-k-1}(\delta_m)$ by Remark 9.7 and Lemma 8.3. Thus we only need to check that $\sigma_k \subset B_{m-k}(\delta_m)$.

To verify this, we prove that $\sigma_{k+1} \subset B_{m-(k+1)}(\delta_m)$. If the layer $k+1$ is thin then this follows from Lemma 9.15(i). If the layer $k+1$ is thick, then denote by $(i, j)$ the thick interval containing $k+1$. By Lemma 10.3 we have $\sigma_{k+1} \subset \mathcal{S}(v_{k+1} x_{k+1})$ ($x_{k+1}$ as in Lemma 10.3). Thus it is enough to establish the inclusion $\mathcal{S}(v_{k+1} x_{k+1}) \subset B_{m-(k+1)}(\delta)$. If $m < j$, then this follows from our assumptions. If $j \leq m$, then from Remark 9.13 and Lemma 9.15(i) we have

$$\mathcal{S}(v_{k+1}x_{k+1}) \subset \mathcal{S}(B_{j-(k+1)}(v_j)) \subset B_{j-(k+1)}(\mathcal{S}(v_j)) \subset \\ \subset B_{j-(k+1)}(\delta_j) \subset B_{m-(k+1)}(\delta_m),$$

as desired.

Hence the projection of $\sigma_k$ onto $B_{m-(k+1)}(\delta_m)$ is defined. Denote it by $\pi$. Since $B_{m-(k+1)}(\delta_m) \subset B_{n-(k+1)}(\tau)$, we have $\pi \subset \sigma_{k+1}$. For the converse inclusion we need $\sigma_{k+1} \subset B_{m-(k+1)}(\delta_m)$, which we have just proved. □

The next lemma is valid for any $(\sigma_k), (\tau_k)$, not necessarily directed geodesics.

**Lemma 10.5.** *Let $e$ be an edge in the layer $k$ of $\Delta$ (between $v_iw_i, v_jw_j$), such that $e$ has three neighboring vertices in the layer $k+1$. Let $\overline{x}$ be a vertex in the residue of $S(e)$ (for some characteristic surface $S$) in the layer $k+1$ between $\sigma, \tau$ in $X$. Then $\overline{x} \in \mathcal{S}(x)$, where $x$ is the vertex in the layer $k+1$ of $\Delta$ in the residue of $e$.*

**Proof.** Denote by $y_1, y_2$ the neighbors of $e$ in the layer $k+1$ of $\Delta$ different from $x$, and let $\overline{y}_1 = S(y_1), \overline{y}_2 = S(y_2)$. We claim that $\overline{y}_1, \overline{y}_2$ are neighbors of $\overline{x}$. Indeed, let $z_1$ be the vertex in $e$, which is a neighbor of $y_1$. Let $\overline{z}_1 = S(z_1) \subset S(e)$. Observe that both $\overline{y}_1, \overline{x}$ lie in the projection of $\overline{z}_1$ onto $B_{n-(k+1)}(\tau)$ (by Remark 9.7), hence, by Lemma 2.8, they are neighbors, as desired. Analogically, $\overline{y}_2, \overline{x}$ are neighbors. Thus, by the easy case of Proposition 7.6, $\overline{x} \in \mathcal{S}(x)$, as required. □

The following lemma describes the behavior of the simplices $\alpha_k$ appearing in the statement of Proposition 10.2. The proof of Lemma 10.6 requires Lemma 9.16(i,ii), apart from this it is straightforward and we skip it. For the same reason we will usually not invoke it in the proof of Proposition 10.2.

**Lemma 10.6.** *Let $\Delta$ be a characteristic disc for some thick interval $(i, j)$. Suppose for some $i \leq l < m \leq j$ we have simplices $\alpha, \alpha'$ in the layers $l, m$ respectively between $v_iw_i, v_jw_j$ in $\Delta$. Suppose that $\alpha \subset S_{m-l}(\alpha')$ and $\alpha' \subset S_{m-l}(\alpha)$. Moreover, assume that $\alpha$ is an interior vertex of $\Delta$ or an edge disjoint with the boundary or $\alpha = v_i$. Assume that $\alpha'$ is an interior vertex or an edge disjoint with boundary or $\alpha' = v_j$. Let $(\alpha_k)_{k=l}^m$ be the directed geodesic in $\Delta$ joining $\alpha$ to $\alpha'$ (in particular $\alpha_l = \alpha, \alpha_m \subset \alpha'$). Then:*

*(i) If $\alpha_k$ is an edge, then $\alpha_{k+1}$ is the unique vertex, which is in the residue of $\alpha_k$ in the layer $k$.*

*(ii) If $\alpha_k = v_k$ and the defect at $v_k$ equals 0, then $\alpha_{k+1} = v_{k+1}$.*

*(iii) If $\alpha_k$ is a vertex with two neighbors in the layer $k+1$, both at distance $m-(k+1)$ from $\alpha'$, then $\alpha_{k+1}$ is an edge spanned by these two vertices.*

*(iv) If $\alpha_k$ is a vertex with two neighbors in the layer $k+1$, but only one of them at distance $m-(k+1)$ from $\alpha'$, then $\alpha_{k+1}$ is this special vertex.*

*Moreover, $\alpha_k$ never equals $w_k$. If $\alpha_k$ is an edge containing $w_k$ then the defect at $w_k$ is $-1$. If $\alpha_k = v_k$, then the defect at $v_k$ is not equal to 1, except possibly for the cases $k = i, j$.*

Now we are ready for the following.

**Proof of Proposition 10.2.** We will prove by induction on $k$, for $l \le k \le m$, the following statement, which, by Lemma 10.6 and Lemma 10.3, implies the proposition.

**Induction hypothesis.** (1) If the layer $k$ is thick and $\alpha_k$ is an edge disjoint with the boundary or meeting the boundary at a vertex of defect $\neq 1$, then $\mathcal{S}(\alpha_k)$ is contained in $\tilde{\sigma}_k$,
(2) if the layer $k$ is thick and $\alpha_k$ is a non–boundary vertex, then $\mathcal{S}(\alpha_k)$ contains $\tilde{\sigma}_k$,
(3) if the layer $k$ is thick and $\alpha_k$ is a boundary vertex or an edge intersecting the boundary at a vertex of defect 1, or the layer $k$ is thin, then $\sigma_k$ contains or is contained in $\tilde{\sigma}_k$.

For $k = l$ the hypothesis is obvious. Suppose it is already proved for some $l \le k \le m-1$. We would like to prove it for $k+1$. First suppose that the layer $k$ is thick and $\alpha_k$ is an edge disjoint with the boundary or meeting the boundary at a vertex of defect $\neq 1$ (case (1)). Then $\alpha_{k+1}$ is a vertex. If it is a boundary vertex, then $v_k \in \alpha_k$. By the induction hypothesis, since the defect at $v_k$ is not 1, $\mathcal{S}(\alpha_k) \subset \tilde{\sigma}_k$, moreover, by Lemma 10.3 we have $\sigma_k \subset \mathcal{S}(\alpha_k)$, hence $\sigma_k \subset \tilde{\sigma}_k$. Hence, by Lemma 2.10, $\tilde{\sigma}_{k+1}$ is contained in the projection of $\sigma_k$ onto $B_{m-(k+1)}(\delta_m)$, which in this case equals $\sigma_{k+1}$ by Lemma 10.4. Thus $\tilde{\sigma}_{k+1} \subset \sigma_{k+1}$, as desired.

Now, still assuming that the layer $k$ is thick and that $\alpha_k$ is an edge disjoint with the boundary or meeting the boundary at a vertex of defect $\neq 1$, suppose that $\alpha_{k+1}$ is not a boundary vertex. Let $\overline{x}$ be any vertex in $\tilde{\sigma}_{k+1}$. Our goal is to prove that $\overline{x} \in \mathcal{S}(\alpha_{k+1})$. By induction hypothesis we know that $\mathcal{S}(\alpha_k) \subset \tilde{\sigma}_k$. Since $\overline{x}$ lies in the layer $k+1$ between $\sigma, \tau$, by Lemma 9.7, we can apply Lemma 10.5 with $e = \alpha_k$. Hence we get $\overline{x} \in \mathcal{S}(\alpha_{k+1})$, as desired.

Thus we have completed the induction step in case (1), i.e. for the layer $k$ thick and $\alpha_k$ an edge disjoint with the boundary or meeting the boundary at a vertex of defect $\neq 1$.

Now suppose that the layer $k$ is thick and $\alpha_k$ is a non–boundary vertex (case (2)). Then it has two neighbors in the layer $k+1$ of $\Delta$, suppose first that both of them are at distance $m-(k+1)$ from $\rho_m$ (we put $\rho_m = v_j$ if $m \geq j$). Then $\alpha_{k+1}$ is the edge spanned by those two vertices. If it intersects the boundary, the defect at the boundary vertex is not 1. Thus we must show that $\mathcal{S}(\alpha_{k+1})$ is contained in $\tilde{\sigma}_{k+1}$. But by induction hypothesis we know that $\mathcal{S}(\alpha_k)$ contains $\tilde{\sigma}_k$. Thus, by Lemma 2.10, it is enough to observe that $\mathcal{S}(\alpha_{k+1}) \subset B_{m-(k+1)}(\delta_m)$. This follows from $\alpha_{k+1} \subset B_{m-(k+1)}(\rho_m)$.

If one of the two neighbors of $\alpha_k$ in the layer $k+1$ is not at distance $m-(k+1)$ from $\rho_m$, then $\alpha_{k+1}$ is the second neighbor, it is a non–boundary vertex (unless $k+1=j$, which will be considered in a moment) and $m<j$. Thus we must show that $\mathcal{S}(\alpha_{k+1})$ contains $\tilde{\sigma}_{k+1}$. Let $\bar{z}$ be a vertex in $\tilde{\sigma}_{k+1}$. Then $\bar{z}$ lies on a 1–skeleton geodesic $\overline{\gamma}$ of length $m-k$ from some vertex of $\tilde{\sigma_k} \subset \mathcal{S}(\alpha_k)$ to some vertex $\bar{x} \in \delta_m = \mathcal{S}(\rho_m)$. We claim that if $\rho_m$ is an edge, then the vertex $x = \mathcal{S}^{-1}(\bar{x}) \in \Delta$ is the vertex closer to $v_m$ then the other vertex of $\rho_m$. Indeed, let $y \in \rho_m$ be the vertex closer to $w_m$. Since $|\alpha_k y| > m-k$ and this distance is realized by a neat geodesic, hence by Lemma 9.8(i) we have $|\mathcal{S}(\alpha_k), \mathcal{S}(y)| > m-k$. This proves the claim. Thus we can apply Lemma 7.6 to $\gamma = \alpha_k x$ and $\bar{z} \in \overline{\gamma}$, and get $\bar{z} \in \mathcal{S}(\alpha_{k+1})$, as desired.

Now we come back to the case $k+1=j$ and $\alpha_k$ a non–boundary vertex. By induction hypothesis we have $\tilde{\sigma}_k \subset \mathcal{S}(\alpha_k)$. By Lemma 9.15(i) we have that $\sigma_{k+1} = \mathcal{S}(v_{k+1})$ (Remark 9.13) lies in $B_{m-(k+1)}(\delta_m)$. Hence, by Lemma 2.10, we have that $\tilde{\sigma}_{k+1}$ contains $\sigma_{k+1}$, as desired.

Thus we have completed the induction step in case (2), i.e. for the layer $k$ thick and $\alpha_k$ a non–boundary vertex.

Now consider the case that the layer $k$ is thick and $\alpha_k$ is a boundary vertex or the layer $k$ is thin, but the layer $k+1$ is thick (in this case put $i=k$). In both cases $\alpha_k = v_k$. First consider the case that the defect at $v_k$ is $-1$ or the layer $k$ is thin. If the hypothesis of Lemma 10.4 are not satisfied, then we can finish as in the previous case (no matter what is the direction of the inclusion given by the induction hypothesis) getting $\tilde{\sigma}_{k+1} \subset \mathcal{S}(\alpha_{k+1})$. Otherwise, $\alpha_{k+1}$ is the edge spanned by two neighbors of $v_k$ in the layer $k+1$. By Lemma 9.16(i,ii) the defect at $v_{k+1}$ equals 1. Hence we want to prove that $\sigma_{k+1}$ either contains or is contained in $\tilde{\sigma}_{k+1}$. We know, by the induction hypothesis, that $\sigma_k$ contains or is contained in $\tilde{\sigma}_k$, hence it is enough to use Lemma 2.10 and Lemma 10.4.

Now assume that either the layer $k$ is thick and $\alpha_k$ is a boundary vertex of defect 0 or an edge intersecting the boundary at a vertex of defect 1, or the layer $k$ is thin and the layer $k+1$ is also thin. Similarly, as before, we have that that $\sigma_k$ contains or is contained in $\tilde{\sigma}_k$ and we want to prove that $\sigma_{k+1}$ contains or is contained in $\tilde{\sigma}_{k+1}$. This follows from Lemma 2.10 and Lemma 10.4.

Thus we have exhausted all the possibilities for case (3) and completed the induction step. □

# 11 Euclidean geodesics between simplices of Euclidean geodesics

In this section we complete the proof of Theorem 10.1. Its first ingredient is Proposition 10.2, proved in section 10. The second ingredient is easy 2–dimensional Euclidean geometry, which we present as a series of lemmas in this section. Throughout the section, we will be treating characteristic discs simultaneously as simplicial complexes and $CAT(0)$ metric spaces.

We start with extending in various ways the notion of a characteristic disc and surface.

**Definition 11.1.** A *generalized characteristic disc* $\Delta$ for an interval $(i, j)$, where $i < j$, is a closed $CAT(0)$ (i.e. simply connected) subspace of $\mathbb{E}^2$ with the following properties. Its boundary is a piecewise linear loop with vertices $v_i, \ldots, v_j, w_j, \ldots, w_i, v_i$ (possibly $v_k = w_k$) , such that for $i \leq k \leq j$ the straight line segments (or points) $v_k w_k$ are contained in consecutive parallel lines at distance $\frac{\sqrt{3}}{2}$. We also require, if $\mathbb{E}^2$ is oriented so that $v_k w_k$ are horizontal, that $v_k$ lies to the left of $w_k$, or $v_k = w_k$.

A *restriction* of a generalized characteristic disc to the interval $(l, m)$, where $i \leq l < m \leq j$, is the generalized characteristic disc enclosed by the loop $v_l \ldots v_m w_m \ldots w_l v_l$. We will denote it by $\Delta|_l^m$. If a generalized characteristic disc comes from equipping a systolic 2–complex with the standard piecewise Euclidean metric, then we call it a *simplicial generalized characteristic disc.*

**Remark 11.2.** Characteristic discs (resp. modified characteristic discs, c.f. Definition 9.10) with the standard piecewise Euclidean metric are simplicial generalized characteristic discs (resp. generalized characteristic discs).

**Definition 11.3.** Suppose we have simplices $(\sigma_k), (\tau_k)$ in the layer $k$ between $\sigma, \tau$ (not necessarily the simplices of the directed geodesics) defined only for $0 \leq i \leq k \leq j \leq n$, where $i < j$, and for $i \leq k < j$ we have that $\sigma_k, \sigma_{k+1}$ span a simplex and $\tau_k, \tau_{k+1}$ span a simplex. Suppose that for $i \leq k \leq j$ the maximal distance between vertices in $\sigma_k$ and in $\tau_k$ is $\geq 2$. Then we define a *partial characteristic disc* and a *partial characteristic surface* in the following way.

Extend $(\sigma_k), (\tau_k)$ to all $0 \leq k \leq n$ so that for $\sigma_k, \sigma_{k+1}$ and $\tau_k, \tau_{k+1}$ span simplices for $0 \leq k < n$, and $\sigma_0, \tau_0 \subset \sigma, \sigma_n, \tau_n \subset \tau$. (This is possible, since, by example, we may issue directed geodesics from $\sigma_i, \tau_i$ to $\sigma$ and from $\sigma_j, \tau_j$ to $\tau$.) Obviously, $\sigma_k, \tau_k$, lie in the layer $k$ between $\sigma, \tau$ for all $0 \leq k \leq n$. Let $(i_{ext}, j_{ext})$ be the thick interval for extended $(\sigma_k), (\tau_k)$ containing $(i, j)$. Let $S\colon \Delta \to X$ be a characteristic surface for $(i_{ext}, j_{ext})$. Then we call $\Delta_{res} = \Delta|_i^j$ a partial characteristic disc (which is a simplicial generalized characteristic disc) and $S_{res} = S|_{\Delta_{res}}$ a partial characteristic surface.

**Caution.** A characteristic surface $S\colon \Delta \to X$, where $\Delta$ is a characteristic disc for a thick interval $(i, j)$ for $(\sigma_k)_{k=0}^n, (\tau_k)_{k=0}^n$ (as in Definition 9.4) is not a partial characteristic surface for $(\sigma_k)_{k=i}^j, (\tau_k)_{k=i}^j$. This is because the layers $i, j$ are thin. But if $i + 1 < j - 1$, then already $S$ restricted to $\Delta|_{i+1}^{j-1}$ is a partial characteristic surface.

Next we show that partial characteristic surfaces satisfy most of the properties of characteristic surfaces. Fix an interval $(i, j)$ and simplices $(\sigma_k)_{k=i}^j, (\tau_k)_{k=i}^j$ as in Definition 11.3. Let $S_{res}\colon \Delta|_{res} \to X$ be a partial characteristic surface, as above.

**Lemma 11.4.**

*(i) $\Delta_{res}$ (and thus $S_{res}$) is flat,*

*(ii) if we embed $\Delta_{res} \subset \mathbb{E}^2_\Delta$, then $v_i w_i$ and $v_j w_j$ are parallel and the consecutive layers between them are contained in consecutive straight lines parallel to $v_i w_i$ and $v_j w_j$.*

*(iii) $S_{res}$ is an isometric embedding on 1–skeleton of a subcomplex spanned by any pair of consecutive layers between $v_i w_i$ and $v_j w_j$ in $\Delta_{res}$.*

*(iv) $\Delta_{res} \subset \mathbb{E}^2_\Delta$ does not depend on the choice of $\sigma_k, \tau_k$ for $k < i$ and $k > j$, the choice of $s_k, t_k$ for $0 \leq k \leq n$, and the choice of $S$.*

*If we have two partial characteristic surfaces $S_1\colon \Delta_1 \to X, S_2\colon \Delta_2 \to X$, then after identifying partial characteristic discs $\Delta_1 = \Delta_2$ (which is possible by (ii)) we have that*

*(v) for any vertices $x, y \in \Delta_1 = \Delta_2$ at distance $1$, $S_1(x)$ and $S_2(y)$ are also at distance $1$,*

*(vi) for any vertex $x \in \Delta_1 = \Delta_2$, $S_1(x)$ and $S_2(x)$ are at distance $\leq 1$.*

*(vii) $S(v_k w_k)$ lies in the layer $k$ between $\sigma$ and $\tau$.*

**Proof.** Assertions (i) and (ii) follow immediately from Lemma 9.6(i,ii). Assertion (iii) follows from Lemma 9.8(i). To prove (iv) notice that $\Delta_{res} = \Delta|_i^j$ is determined by the distances $|s_k t_k|$ for $i \leq k \leq j$ and $|s_k t_{k+1}|$ for $i \leq k < j$, by (iii). Hence, if we fix $s_k$ and $t_k$ for $i \leq k \leq j$, then $\Delta_{res}$ does not depend on the extension of $(\sigma_k)_{k=i}^j, (\tau_k)_{k=i}^j$. On the other hand, if we fix such an extension, then $|s_k t_k|, |s_k t_{k+1}|$ do not depend on the choice of $s_k, t_k$, by Lemma 9.8(ii).

It is a bit awkward to try to obtain assertion (v) as a consequence of Lemma 9.8(iii). Let us say, instead, that assertion (v) follows immediately from the proof of Lemma 9.8(iii). Similarly, assertion (vi) follows from the proof of Lemma 9.8(iv). Assertion (vii) follows directly from Remark 9.7.
□

**Definition 11.5.** We define the *partial characteristic image* $\mathcal{S}(\rho)$ of a simplex $\rho$ in the partial characteristic disc as the span of $S(\rho)$ over all partial characteristic surfaces $S$. By Lemma 11.4(v,vi), $\mathcal{S}(\rho)$ is a simplex. We call this assignment the *partial characteristic mapping*. Like in Definition 9.9 we can consider also the assignment $\mathcal{S}^{-1}$.

**Definition 11.6.** Let $\Delta$ be a generalized characteristic disc and $\gamma$, $\gamma'$ be two paths connecting some points on $v_i w_i$ to points on $v_j w_j$ such that intersections of $\gamma$, $\gamma'$ with $v_k w_k$ are unique for each $i \leq k \leq j$. We say that $\gamma, \gamma'$ are *$d$–close* if they intersect $v_k w_k$ in points at distance $\leq d$ for each $i \leq k \leq j$.

The following lemma describes the possible displacements of $CAT(0)$ geodesics in characteristic discs when perturbing the boundary and the endpoints.

**Lemma 11.7.** *Let $\Delta' \subset \Delta$ be two generalized characteristic discs for $(i, j)$ such that for each $i \leq k \leq j$ we have $v'_k w'_k \subset v_k w_k$ (and the order is $v_k v'_k w'_k w_k$) and $|v_k v'_k| \leq d$, $|w_k w'_k| \leq d$. Then for any points $x \in v_i w_i$, $y \in v_j w_j$, $x' \in v'_i w'_i$, $y' \in v'_j w'_j$ such that $|xx'| \leq d$, $|yy'| \leq d$, the $CAT(0)$ geodesics from $x$ to $y$ in $\Delta$ and from $x'$ to $y'$ in $\Delta'$ are $d$–close in $\Delta$.*

**Proof.** Denote by $\gamma, \gamma'$ the geodesics from $x$ to $y$ in $\Delta$ and from $x'$ to $y'$ in $\Delta'$ respectively. Denote by $N_d(\gamma)$ the set of points in $\Delta$ at distance $\leq d$ from $\gamma$ in the direction parallel to $v_k w_k$ (i.e. the intersection with $\Delta$ of the union of translates of $\gamma$ by a distance $\leq d$ in the direction parallel to $v_k w_k$), and by $N'_d(\gamma)$ the intersection $N_d(\gamma) \cap \Delta'$.

Observe that $N'_d(\gamma)$ is connected, since for each $k$ the set $v'_k w'_k \cap N'_d(\gamma)$ is nonempty and the intersection of $N'_d(\gamma)$ with each of the parallelograms $v'_k w'_k w'_{k+1} v'_{k+1}$ is an intersection of two parallelograms, hence convex and connected. We claim that $N_d(\gamma)$ is convex in $\Delta$. To establish this, we need to study the interior angle at vertices of $\partial N_d(\gamma)$ outside $\partial\Delta$. The only possibility for angle $> 180°$ is at the horizontal translates of break points of $\gamma$. But since $\gamma$ is a $CAT(0)$ geodesic, then each of its break points lies on the boundary of $\Delta$, and the translate, for which possibly the angle is $> 180°$, lies outside $\Delta$. Thus the claim follows. Hence (by connectedness) $N'_d(\gamma)$ is convex in $\Delta'$. Thus $\gamma' \subset N'_d(\gamma)$ and we are done. □

Let us prepare the setting for the next lemma. It will help us deal with the data given by Proposition 10.2, which is, roughly speaking, a pair of surfaces spanned on nearby pairs of geodesics. To be more precise, let $\hat{\sigma}_k, \hat{\tau}_k, \tilde{\sigma}_k, \tilde{\tau}_k$ be simplices in the layers $i \leq k \leq j$ between $\sigma, \tau$ satisfying conditions of Definition 11.3. Moreover, assume that for each $i \leq k \leq j$ we have that $\hat{\sigma}_k \subset \tilde{\sigma}_k$ or $\tilde{\sigma}_k \subset \hat{\sigma}_k$, and $\hat{\tau}_k \subset \tilde{\tau}_k$ or $\tilde{\tau}_k \subset \hat{\tau}_k$. Let $\hat{\Delta}, \tilde{\Delta}$ be associated partial characteristic discs, unique by 11.4(iv). Denote the boundary vertices of $\hat{\Delta}$ (resp. $\tilde{\Delta}$) by $\hat{v}_k, \hat{w}_k$ (resp. $\tilde{v}_k, \tilde{w}_k$), its characteristic mapping by $\hat{\mathcal{S}}$ (resp. $\tilde{\mathcal{S}}$).

**Lemma 11.8.** *There exists a simplicial generalized characteristic disc $\overline{\Delta}$ for $(i, j)$ and embeddings (thought of as inclusions, for simplicity) $\overline{\Delta} \subset \hat{\Delta}, \overline{\Delta} \subset \tilde{\Delta}$ such that the distances $|\overline{v}_k\hat{v}_k|$, $|\overline{w}_k\hat{w}_k|$ in $\hat{\Delta}$ and distances $|\overline{v}_k\tilde{v}_k|, |\overline{w}_k\tilde{w}_k|$ in $\tilde{\Delta}$ are all $\leq 1$ for $i \leq k \leq j$. Moreover, $|\overline{v}_k\overline{w}_k| \geq 1$ for $i \leq k \leq j$.*

**Proof.** For each $i \leq k \leq j$, let $\sigma_k^{max}$ be the greater among $\hat{\sigma_k}, \tilde{\sigma_k}$ and let $\sigma_k^{min}$ be the smaller, let $\tau_k^{max}$ be the greater among $\hat{\tau_k}, \tilde{\tau_k}$ and let $\tau_k^{min}$ be the smaller. Pick vertices $x_k \in \sigma_k^{max}, y_k \in \tau_k^{max}$ so that the distance $|x_k y_k|$ is maximal. If possible, choose them from $\sigma_k^{min}, \tau_k^{min}$ (if it is possible for $x_k, y_k$ independently, then it is possible for both of them at the same time, by Lemma 9.5). Pick a 1–skeleton geodesic $\phi_k$ connecting $x_k$ to $y_k$ intersecting $\sigma_k^{min}, \tau_k^{min}$ (this is possible by Corollary 8.7). If $x_k \in \sigma_k^{min}$, then put $\overline{s}_k = x_k$, otherwise let $\overline{s}_k$ be the neighbor of $x_k$ on $\phi_k$. Analogically, if $y_k \in \tau_k^{min}$, then put $\overline{t}_k = y_k$, otherwise let $\overline{t}_k$ be the neighbor vertex of $y_k$ on $\phi_k$. Thus $\overline{s}_k \in \sigma_k^{min}, \overline{t}_k \in \tau_k^{min}$. Let $\overline{\Delta}$ be the partial characteristic disc for $(\overline{s}_k), (\overline{t}_k)$ for $i \leq k \leq j$. Denote its boundary vertices by $\overline{v}_k, \overline{w}_k$.

The embedding, say $\overline{\Delta} \subset \hat{\Delta}$, is defined as follows. By Proposition 7.6 there exists a characteristic surface $\overline{S}\colon \overline{\Delta} \to X$ such that $\overline{S}(\overline{v}_k\overline{w}_k) \subset \phi_k$. Moreover, again by Proposition 7.6, the sub–geodesic $\overline{s}_k\overline{t}_k$ of $\phi_k$ lies in $\hat{\mathcal{S}}(\hat{\Delta})$. Hence we can define the desired mapping as the composition $\hat{\mathcal{S}}^{-1} \circ \overline{S}$. To check that this is an embedding it is enough to check that it preserves the the layers (Lemma 11.4(vii)) and is isometric on the layers (Lemma 11.4(iii)).

To prove the last assertion fix $k$ and assume w.l.o.g. that $\sigma_k^{min} = \hat{\sigma_k}$. Then $|\overline{v}_k\overline{w}_k| \geq |\hat{v}_k\hat{w}_k| - 1 \geq 1$, as desired. □

Now let us prepare the statement of our final lemma. One can view it as a simple case of Theorem 10.1, case of $X$ being flat.

Let $\Delta$ be a characteristic disc for a thick interval $(i, j)$ for the directed geodesics $(\sigma_k), (\tau_k)$ between $\sigma, \tau$ and let $\gamma'$ be its $CAT(0)$ diagonal, c.f. Definition 9.10. Let $(\rho_k)_{k=i+1}^{j-1}$ be simplices of the Euclidean diagonal in $\Delta$ (Definition 9.10). Fix $i \leq l < m \leq j$. If $i < l < m < j$ then let $(\alpha_k)_{k=l}^{m}, (\beta_k)_{k=m}^{l}$ be directed geodesics in $\Delta$ from $\rho_l$ to $\rho_m$ and from $\rho_m$ to $\rho_l$ respectively. If $l = i$ then put $\rho_i = v_i$ in the definition of $(\alpha_k)_{k=l}^{m}$ and $\rho_i = w_i$ in the definition of $(\beta_k)_{k=m}^{l}$. If $m = j$ then put $\rho_j = w_j$ in the definition of $(\beta_k)_{k=m}^{l}$ and $\rho_j = v_j$ in the definition of $(\alpha_k)_{k=l}^{m}$. For all other purposes we will put $\rho_i = v_i w_i, \rho_j = v_j w_j$.

Let $\bigcup \hat{\Delta}$ be the subcomplex of $\Delta$ which is the span of the union of $\mathrm{conv}\{\alpha_k, \beta_k\}$ over all $l \leq k \leq m$. Note that $\bigcup \hat{\Delta}$ is a simplicial generalized characteristic disc. Denote the vertices of its boundary loop by $(\hat{v}_k)$ and $(\hat{w}_k)$. Denote by $\hat{\gamma}$ the $CAT(0)$ geodesic joining in $\bigcup \hat{\Delta}$ the barycenters of $\rho_l$ and $\rho_m$ (which lie in in $\bigcup \hat{\Delta}$).

**Lemma 11.9.** *$\gamma'$ restricted to $\Delta|_l^m$ and $\hat{\gamma}$ are $\frac{1}{2}$–close in $\Delta|_l^m$ .*

**Proof.** Let us denote by $\bigcup \hat{\Delta}_0$ the generalized characteristic disc obtained from $\bigcup \hat{\Delta}$ by removing the following triangles: For any boundary vertex of defect 1 in the layers $\neq l, m$, say $\hat{v}_k$, cut off a triangle along the segment $\hat{v}_{k-1}\hat{v}_{k+1}$. For any boundary vertex of defect 2 (which is possible in the layers $l, m$), say $\hat{v}_l$, cut off a triangle joining $\hat{v}_{l+1}$ to the barycenter of $\hat{v}_l\hat{w}_l$.

We claim that $\bigcup \hat{\Delta}_0$ is convex in $\Delta$ (treated as $CAT(0)$ spaces). This means that at all vertices of $\partial \bigcup \hat{\Delta}_0$ outside $\partial\Delta$, the interior angle of $\bigcup \hat{\Delta}_0$ is $\leq 180°$. We skip the proof, which is an easy consequence of Lemma 10.6.

Let $\hat{\gamma}_0$ be the $CAT(0)$ geodesic in $\bigcup \hat{\Delta}_0$ joining the barycenter $\hat{x}$ of $\rho_l$ with the barycenter $\hat{y}$ of $\rho_m$ (observe that $\hat{x}, \hat{y} \in \bigcup \hat{\Delta}_0$). Since $\bigcup \hat{\Delta}_0 \subset \Delta$ is convex, $\hat{\gamma}_0$ agrees with the $CAT(0)$ geodesic in $\Delta$ joining $\hat{x}, \hat{y}$.

Now we apply Lemma 11.7 to $\Delta'|_l^m \subset \Delta|_l^m$ (c.f. Definition 9.10 for the definition of $\Delta'$), and geodesics $\hat{\gamma}_0$ in $\Delta|_l^m$ and $\gamma'$ restricted to $\Delta'|_l^m$. Observe that endpoints $\hat{x}, \hat{y}$ of $\hat{\gamma}_0$ are at distance $\leq \frac{1}{2}$ from $\gamma' \cap v_l w_l, \gamma' \cap v_m w_m$ by the definition of $\rho_l, \rho_m$. Hence, by Lemma 11.7, we have that $\hat{\gamma}_0$ is $\frac{1}{2}$–close to $\gamma'$ restricted to $\Delta|_l^m$.

Now observe that since $\bigcup \hat{\Delta}_0$ is also convex in $\bigcup \hat{\Delta}$, we have $\hat{\gamma}_0 = \hat{\gamma}$ and we are done. □

Finally, we can proceed with the following.

**Proof of Theorem 10.1.** First suppose that the layer $k$ for $(\sigma_t), (\tau_t)$ is thin. Then, by Proposition 10.2(i), $\tilde{\sigma}_k$ contains or is contained in $\sigma_k$ and $\tilde{\tau}_k$

contains or is contained in $\tau_k$. Hence the thickness of the layer $k$ for $(\tilde{\sigma}_t),(\tilde{\tau}_t)$ is $\leq 3$ and thus $\tilde{\sigma_k} \subset B_1(\tilde{\delta}_k)$ or $\tilde{\tau_k} \subset B_1(\tilde{\delta}_k)$, hence $|\tilde{\delta}_k, \delta_k| \leq 1$.

Now suppose that the layer $k$ for $(\sigma_t),(\tau_t)$ is thick and suppose it is contained in a thick interval $(i,j)$ with a characteristic disc $\Delta$. Put $\rho_l = v_i w_i$ if $l \leq i$ and $\rho_m = v_j w_j$ if $m \geq j$. We will use the notation introduced before Lemma 11.9. First suppose that the layer $k$ for $(\tilde{\sigma}_t),(\tilde{\tau}_t)$ is thin. Then, by Proposition 10.2(ii), the maximal distance between vertices in $\mathcal{S}(\alpha_k)$ and $\mathcal{S}(\beta_k)$, hence in $\alpha_k$ and $\beta_k$ is $\leq 3$. Since $\hat{\gamma} \cap v_k w_k$ lies in $\mathrm{conv}\{\alpha_k, \beta_k\}$, Lemma 11.9 implies that $\gamma' \cap v_k w_k$ is at distance $\leq \frac{1}{2}$ from $\mathrm{conv}\{\alpha_k, \beta_k\}$. Hence $\alpha_k \subset B_1(\rho_k)$ or $\beta_k \subset B_1(\rho_k)$. Thus $\tilde{\delta}_k, \delta_k$ are at distance $\leq 1$.

Now suppose that the layer $k$ for $(\tilde{\sigma}_t),(\tilde{\tau}_t)$ is thick. Let $\tilde{\Delta}$ be the characteristic disc for the thick interval $(\tilde{i},\tilde{j})$ containing $k$ for $(\tilde{\sigma}_t),(\tilde{\tau}_t)$. If the layer $k$ for $(\alpha_t),(\beta_t)$ (between $\rho_l, \rho_m$ in $\Delta$) is thin, then the thickness of the layer $k$ for $(\tilde{\sigma}_t),(\tilde{\tau}_t)$ is $\leq 3$, by Proposition 10.2(ii). Hence $\tilde{\sigma}_k \subset B_1(\tilde{\delta}_k)$ or $\tilde{\tau}_k \subset B_1(\tilde{\delta}_k)$. By Lemma 11.9 we have $|\rho_k, \alpha_k| \leq 1$ and $|\rho_k, \beta_k| \leq 1$, hence altogether $|\tilde{\delta}_k, \delta_k| \leq 2$.

So suppose that the layer $k$ for $(\alpha_t),(\beta_t)$ in $\Delta$ is thick, let $\hat{i},\hat{j}$ be the thick interval for $(\alpha_t),(\beta_t)$ containing $k$ and $\hat{\Delta}$ the corresponding characteristic disc. Observe that $\hat{\Delta} = \bigcup \hat{\Delta}|_{\hat{i}}^{\hat{j}}$. Let $i_{max}$ be the maximum of $\hat{i},\tilde{i}$ and $j_{min}$ be the minimum of $\hat{j},\tilde{j}$. Obviously $i_{max} < k < j_{min}$. Assume $i_{max}+1 < j_{min}-1$, in the case of equality the argument is similar and we omit it.

By Proposition 10.2(ii) we can apply Lemma 11.8 to $\tilde{\Delta}$ and $\hat{\Delta}$ restricted to $(i_{max}+1, j_{min}-1)$. Denote by $\overline{\Delta}$ the simplicial generalized characteristic disc for $(i_{max}+1, j_{min}-1)$ guaranteed by Lemma 11.8. Denote by $\overline{\Delta}'$ the generalized characteristic disc obtained from $\overline{\Delta}$ by removing horizontal (the direction of $\overline{v}_t\overline{w}_t$) $\frac{1}{2}$–neighborhood of the boundary, which is allowed since $|\overline{vw}| \geq 1$ by Lemma 11.8. Let $\tilde{\Delta}'$ be the modified characteristic in $\tilde{\Delta}$ and $\tilde{\gamma}'$ the $CAT(0)$ diagonal of $\tilde{\Delta}$ (c.f. Definition 9.10). Define a generalized characteristic disc $\hat{\Delta}'$ and a $CAT(0)$ geodesic $\hat{\gamma}'$ in $\hat{\Delta}'$ in the following way. For each $\hat{i} \leq t \leq \hat{j}$ denote by $\hat{v}'_t, \hat{w}'_t$ points on $\hat{v}_t\hat{w}_t$ at distance $\frac{1}{2}$ from $\hat{v}_t, \hat{w}_t$, respectively, if $\hat{v}_t \neq \hat{w}_t$. Otherwise, put $\hat{v}'_t = \hat{v}_t, \hat{w}'_t = \hat{w}_t$. Let $\hat{\Delta}'$ be the generalized characteristic disc enclosed by the loop $\hat{v}'_{\hat{i}} \ldots \hat{v}'_{\hat{j}}\hat{w}'_{\hat{j}} \ldots \hat{w}'_{\hat{i}}\hat{v}'_{\hat{i}}$. Let $\hat{\gamma}'$ be the $CAT(0)$ geodesic in $\hat{\Delta}'$ joining $\hat{v}'_l = \hat{w}'_l$ and $\hat{v}'_m = \hat{w}'_m$. By Lemma 11.8 we have inclusions of $\overline{\Delta}'$ into $\hat{\Delta}'|_{i_{max}+1}^{j_{min}-1}, \tilde{\Delta}'|_{i_{max}+1}^{j_{min}-1}$ with distances $|\overline{v}'_t\hat{v}'_t|, |\overline{w}'_t\hat{w}'_t|$ in $\hat{\Delta}'$, and distances $|\overline{v}'_t\tilde{v}'_t|, |\overline{w}'_t\tilde{w}'_t|$ in $\tilde{\Delta}'$ all $\leq 1$ for $i_{max}+1 \leq t \leq j_{min}-1$.

Now we will choose a special point $\overline{x} \in \overline{v}'_{i_{max}+1}\overline{w}'_{i_{max}+1}$. W.l.o.g. assume that $i_{max} = \tilde{i}$, hence $|\tilde{v}_{i_{max}+1}\tilde{w}_{i_{max}+1}| = 2$. Choose any $\hat{x}$ in $\overline{v}'_{i_{max}+1}\overline{w}'_{i_{max}+1}$ at distance $\leq 1$ from $\hat{\gamma}'$, which is possible, since $|\overline{v}'_{i_{max}+1}\hat{v}'_{i_{max}+1}| \leq 1$ and $|\overline{w}'_{i_{max}+1}\hat{w}'_{i_{max}+1}| \leq 1$. Since $|\tilde{v}'_{i_{max}+1}\tilde{w}'_{i_{max}+1}| \leq 1$, $\overline{x}$ is also at distance $\leq 1$

from $\tilde{\gamma}'$. Choose $\overline{y}$ in $\overline{v}'_{j_{min}-1}\overline{w}'_{j_{min}-1}$ in an analogous way.

By this construction the endpoints of $\tilde{\gamma}'$ and $\hat{\gamma}'$ restricted to $(i_{max}+1, j_{min}-1)$ are at distance $\leq 1$ from $\bar{x}, \bar{y}$ in $\tilde{\Delta}'|_{i_{max}+1}^{j_{min}-1}, \hat{\Delta}'|_{i_{max}+1}^{j_{min}-1}$, respectively. Thus, using twice Lemma 11.7, we get that $\tilde{\gamma}'$ and $\hat{\gamma}'$ restricted to $(i_{max}+1, j_{min}-1)$ are 1–close to the $CAT(0)$ geodesic $\bar{x}\bar{y}$ in $\overline{\Delta}'$ (in $\tilde{\Delta}'|_{i_{max}+1}^{j_{min}-1}, \hat{\Delta}'|_{i_{max}+1}^{j_{min}-1}$ respectively).

By Lemma 11.9, $\gamma'$ and $\hat{\gamma}$ are $\frac{1}{2}$–close in $\Delta|_l^m$. By Lemma 11.7, $\hat{\gamma}'$ and $\hat{\gamma}$ are $\frac{1}{2}$–close in $\hat{\Delta}|_{\hat{i}}^{\hat{j}}$. Putting those four estimates together we get that $\delta_k, \tilde{\delta}_k$ are at distance $\leq 3$, as desired. □

We end this section by indicating, how Theorem 10.1 can be promoted to Theorem B, with a reasonable constant $C$. The difference in statements comes from substituting $\delta_l, \delta_m$ with $x \in \delta_l,\ y \in \delta_m$ such that $|xy| = m-l$. As a first step, we check that Proposition 10.2 implies that the directed geodesics between $x$ and $y$ lie near the union of characteristic images of characteristic discs for $(\sigma_k), (\tau_k)$. This follows from the fact that directed geodesics in systolic complexes satisfy the so called fellow traveler property with a good constant, see [16]. The second step is to reprove Lemma 11.8 allowing $\hat{\sigma}_k$ and $\tilde{\sigma}_k$ (and similarly $\hat{\tau}_k$ and $\tilde{\tau}_k$) to be farther apart, at distance bounded by the above fellow traveler constant. Then some minor changes in the proof of Theorem 10.1 yield Theorem B.

We will give a different complete proof of Theorem B (though with a worse constant) in the next section.

# 12 Characteristic discs spanned on Euclidean geodesics

In this section we prove the following crucial proposition, which, roughly speaking, says that in a characteristic disc spanned on a Euclidean geodesic and an arbitrary other geodesic, the boundary segment corresponding to the Euclidean geodesic is coarsely a $CAT(0)$ geodesic. We introduce the following notation, which will be fixed for the whole section.

Let $\sigma, \tau$ be simplices in a systolic complex $X$ satisfying as before $\sigma \subset S_n(\tau), \tau \subset S_n(\sigma)$ and suppose $(p_k)_{k=0}^n, (r_k)_{k=0}^n$ are 1–skeleton geodesics with endpoints in $\sigma$ and $\tau$ such that $r_k \in \delta_k$, where $(\delta_k)_{k=0}^n$ is the Euclidean geodesic between $\sigma$ and $\tau$. Let $0 \leq i_{pr} < j_{pr} \leq n$ be a thick interval for $(p_k), (r_k)$ and let $\Delta_{pr}, \mathcal{S}_{pr}$ be the corresponding characteristic disc and mapping. Let $\gamma_{pr}$ be the $CAT(0)$ geodesic in $\Delta_{pr}$ joining the barycenters of the unique edges in the layers $i_{pr}, j_{pr}$.

**Proposition 12.1.** *$\gamma_{pr}$ is 99–close to the boundary path $\mathcal{S}_{pr}^{-1}\big((r_k)\big)$.*

This proposition has fundamental consequences. One of them is Theorem C, which says roughly this: in a "Euclidean geodesic triangle", the distance between the midpoints of two sides is, up to an additive constant, smaller than half of the length of the third side. We study this in the next section.

The second consequence of Proposition 12.1 is an alternative proof of the following.

**Theorem 12.2** (Theorem B)**.** *Let $\sigma, \tau$ be simplices of a systolic complex $X$, such that for some natural $n$ we have $\sigma \subset S_n(\tau), \tau \subset S_n(\sigma)$. Let $(\delta_k)_{k=0}^n$ be the Euclidean geodesic between $\sigma$ and $\tau$. Take some $0 \leq l < m \leq n$ and let $(r_k)_{k=l}^m$ be a 1–skeleton geodesic such that $r_k \in \delta_k$ for $l \leq k \leq m$. Consider the simplices $\tilde{\delta}_l = r_l, \tilde{\delta}_{l+1}, \ldots, \tilde{\delta}_m = r_m$ of the Euclidean geodesic between vertices $r_l$ and $r_m$. Then for each $l \leq k \leq m$ we have $|\delta_k, \tilde{\delta}_k| \leq C$, where $C$ is a universal constant.*

**Proof.** Extend $(r_k)_{k=l}^m$ to a 1–skeleton geodesic $(r_k)_{k=0}^n$ between $\sigma, \tau$ so that $r_k \in \delta_k$ (this is possible by Lemma 9.15(i)). Let $(\tilde{r}_k)_{k=l}^m$ be any 1–skeleton geodesic between $r_l$ and $r_m$ such that $\tilde{r}_k \in \tilde{\delta}_k$. Put additionally $\tilde{r}_k = r_k$ for $0 \leq k < l$ and for $m < k \leq n$. Let $\Delta_{r\tilde{r}}$ be the characteristic disc for some thick interval for $(\tilde{r}_k)_{k=0}^n, (r_k)_{k=0}^n$ and let $\gamma_{r\tilde{r}}$ be the $CAT(0)$ geodesic joining the barycenters of its outermost edges. Let $\mathcal{S}_{r\tilde{r}}$ be the corresponding characteristic mapping.

Notice that $\Delta_{r\tilde{r}}$ is also a characteristic disc for $(r_k)_{k=l}^m, (\tilde{r}_k)_{k=l}^m$ between $r_l$ and $r_m$. Applying twice Proposition 12.1 we obtain that $\gamma_{r\tilde{r}}$ is 99–close to both $\mathcal{S}_{r\tilde{r}}^{-1}\big((r_k)\big)$ and $\mathcal{S}_{r\tilde{r}}^{-1}\big((\tilde{r}_k)\big)$. This proves that for all $l \leq k \leq m$ we have $|r_k \tilde{r}_k| \leq 198$, hence $|\delta_k, \tilde{\delta}_k| \leq 198$. Thus any $C \geq 198$ satisfies the assertion of the theorem. □

The proof of Proposition 12.1 is rather technical. This is the reason we decided to present the straightforward proof of Theorem 10.1 (the weak version of Theorem B) via Proposition 10.2. Before we get into technical details of the proof, split into various lemmas, we present an outline, which hopefully helps to keep track of the main ideas.

**Outline of the proof of Proposition 12.1.** We are dealing with configurations of four geodesics between $\sigma$ and $\tau$: the directed geodesics, which we denote by $(\sigma_k)_{k=0}^n, (\tau_k)_{k=0}^n$, as in the previous sections, $(r_k)_{k=0}^n$, which goes along the Euclidean geodesic $\delta_k$, and the fourth arbitrary 1–skeleton geodesic $(p_k)_{k=0}^n$. For the layer $k$ thick (for $(\sigma_k), (\tau_k)$) we have that $\delta_k = \mathcal{S}(\rho_k)$, where $\rho_k$ is the simplex of the Euclidean diagonal in appropriate characteristic disc $\Delta$ for $(\sigma_k), (\tau_k)$. Hence we need to find out, what is the possible position of $(p_k)$ w.r.t. $\mathcal{S}(\Delta)$. It turns out that in each layer there are 1–skeleton geodesics between simplices $\sigma_k, \tau_k$ and $p_k$, which form a very thin triangle (Lemma 12.3). The intersection with $\mathcal{S}(\Delta)$ of the center simplex of this triangle will be later denoted by $\overline{\chi}_k$.

In Lemma 12.4 we study, how do $\overline{\chi}_k$ vary with $k$. Assume for simplicity that $p_k$ stay away from $\mathcal{S}(\Delta)$. Then it turns out that first (i.e. for small $k$) $\overline{\chi}_k$ follow $\mathcal{S}(w_k)$, next the barycenters of $\overline{\chi}_k$ lie in the characteristic image of a vertical line in $\Delta$ and last $\overline{\chi}_k$ follow $\mathcal{S}(v_k)$. The $CAT(0)$ diagonal $\gamma'$ of $\Delta$ crosses this line at most once. Thus we can divide each "thick" interval (an interval with all layers thick, in opposition to the thick interval with thin endpoint layers) for $(\sigma_k), (\tau_k)$ into three subintervals: the "initial" one, for which $\chi_k = \mathcal{S}^{-1}(\overline{\chi}_k)$ is far to the right from $\rho_k$ or near $w_k \in \partial\Delta$, the "middle" one, for which $\chi_k$ is near $\rho_k$, and the "final" one, for which $\chi_k$ is far to the left from $\rho_k$ or near $v_k \in \partial\Delta$, see Lemma 12.8. Moreover, in the "initial" (resp. "final") interval we can distinguish a "pre–initial" (resp. "post–final") interval in which $\gamma'$ stays away from $w'_k \in \partial\Delta'$ (resp. $v'_k \in \partial\Delta'$), where $\Delta'$ is the modified characteristic disc. This distinction is done in the main body of the proof of Proposition 12.1. The vertices $\mathcal{S}_{pr}^{-1}(r_k)$ in $\Delta_{pr}$, for $k$ in one of these intervals, are positioned as follows. The vertices of the "middle" interval together with the vertices of the other ones outside the "pre–initial" and "post–final" intervals form a coarse vertical line (this is a consequence of Lemma 12.9), while the vertices of the "pre–initial" and "post–final" intervals form also coarse $CAT(0)$ geodesics, fortunately forming with the coarse vertical line angles $\geq 180°$ at the endpoints. This proves Proposition 12.1 in the simple case of a single "thick" interval for $(\sigma_k), (\tau_k)$.

In the complex case, the question is, how may the various "thick" intervals and thin layers for $(\sigma_k), (\tau_k)$ alternate. We define roughly the following notions. A "thin" interval is an interval of not very thick layers. A "proper thin" interval is a "thin" interval with thin layers at the beginning and at the end. A "very thick" interval is an interval containing a layer that is very thick. In Lemma 12.11 we prove that the vertices $\mathcal{S}_{pr}^{-1}(r_k)$, for $k$ in a "thin" interval, form a coarse vertical line. In Corollary 12.10 we prove that if at the beginning of a thin layer there is an adjoined "thick" interval, then this "thick" interval has the "final" subinterval constructed above "thin". Similarly, if at the end of a thin layer there is an adjoined "thick" interval, then this thick interval has the "initial" subinterval "thin". The last piece of the puzzle is an assertion in Lemma 12.8, that for a "very thick" interval, either its "initial" or "final" subinterval is non–"thin".

The way to put these pieces together is the following. We take a maximal "proper thin" interval. The "very thick" interval adjoined at the beginning of this "proper thin" interval must have either the "initial" or the "final" subinterval non–"thin" (Lemma 12.8), but the possibility of the "final" subinterval non–"thin" is excluded (Corollary 12.10). Thus its "initial" subinterval is non–"thin" and this excludes the possibility that some thin layer (hence any layer) is adjoined at the beginning of this "very thick" interval (Corollary 12.10). We can apply analogous considerations to the "very thick" interval adjoined at the end of the "proper thin" interval. Altogether, we have the following configuration: the "proper thin" interval, with a "very thick" interval with "thin" "final" subinterval adjoined at the beginning, and with a "very thick" interval with "thin" "initial" subinterval adjoined at the end. Moreover, in the first of the "very thick" intervals we distinguish the "pre–initial" interval and in the second one we distinguish the "post–final" interval. The vertices $\mathcal{S}_{pr}^{-1}(r_k)$, for $k$ outside the "pre–initial" and "post–final" intervals, form a coarse vertical line (Lemma 12.9 and Lemma 12.11), and the ones for $k$ in the "pre–initial" and "post–final" intervals form also coarse $CAT(0)$ geodesics forming with the coarse vertical line angles $\geq 180°$ at the endpoints. This ends the outline of the proof of Proposition 12.1.

The following lemma treats configurations of three vertices in a layer. Denote the layers between $\sigma, \tau$ by $L_k$.

**Lemma 12.3.** *Suppose $p, s, t$ are three vertices in $L_k$. Then either there exists a vertex such that there are 1–skeleton geodesics $ps, pt, st$ passing through this vertex or there exists a triangle (i.e. a 2–simplex) such that there are 1–skeleton geodesics $ps, pt, st$ passing through the edges of this triangle.*

**Proof.** Let $p'$ be a vertex farthest from $p$ lying both on some 1–skeleton geodesic $ps$ and some 1–skeleton geodesic $pt$. Let $s'$ be a vertex farthest from $s$ lying both on some 1–skeleton geodesic $sp'$ and some 1–skeleton geodesic $st$. Finally let $t'$ be a vertex farthest from $t$ lying both on some 1–skeleton geodesic $tp'$ and some 1–skeleton geodesic $ts'$. If two of the vertices $p', s', t'$ coincide, then all three coincide and the lemma follows immediately. Suppose now that those three vertices are distinct.

From the choice of $p', s', t'$ it follows that any loop $\Gamma$ obtained by concatenating some 1–skeleton geodesics $p's', s't', t'p'$ is embedded in $L_k$. Since $L_k$ is convex (Remark 8.2), it is contractible (see remarks after Definition 2.4), hence $\Gamma$ is contractible in $L_k$ (we could also invoke Lemma 8.4). Consider a surface $T\colon D \to L_k$ of minimal area spanned on such a geodesic triangle $\Gamma$ (we allow the geodesics to vary). By minimality of area the defects at interior vertices of $D$ and at interior vertices of the boundary geodesics are non–positive. Since by Gauss–Bonnet Lemma 7.2 the total sum of defects equals 6, we get that all mentioned vertices have defects 0 and the vertices of the geodesic triangle $D$ have defect 2. Hence $D$ is a subcomplex of $\mathbb{E}^2_\Delta$ which is a Euclidean equilateral triangle. Denote the length of the side of this triangle by $d > 0$. If $d \geq 2$ then let $u$ be the vertex such that $T(u) = p'$, let $u_1, u_2$ be its neighbors in $D$, let $u_3$ be the common neighbor of $u_1, u_2$ in $D$ different from $u$ and let $u_4$ be the neighbor of $u_1$ different from previously mentioned vertices. By Lemma 8.6 applied to the trapezoid $T(u)T(u_1)T(u_2)T(u_3)T(u_4)$ either we have an edge $T(u)T(u_3)$ or $T(u_2)T(u_4)$. In the first case the vertex $T(u_3)$ turns out to lie on some 1–skeleton geodesics $sp, tp$ contradicting the choice of $p'$. In the second case the vertex $T(u_2)$ turns out to lie some 1–skeleton geodesics $sp, tp$, also giving a contradiction. Hence $d = 1$ and the lemma follows. □

In the next lemma we analyze the possible position of $(p_k)$ w.r.t. the partial characteristic image $\mathcal{S}(\Delta)$ of a partial characteristic disc $\Delta$ for $(i, j)$ for $(\sigma_k), (\tau_k)$. This means that we assume that the layers $i \leq k \leq j$ are thick, c.f. Definition 11.3. In the language of the outline of the proof of Proposition 12.1 this is the "thick" interval. The boundary vertices of $\Delta$ are, as always, denoted by $(v_k), (w_k)$.

For each $i \leq k \leq j$ let $s_k \in \sigma_k, t_k \in \tau_k$ be chosen as in the previous sections to maximize the distance $|s_k t_k|$. Moreover, among those, choose $s_k, t_k$ to maximize the distances $|p_k s_k|, |p_k t_k|$ (it is possible to do this independently by Lemma 9.5). For each $k$ perform in $L_k$ the construction of $s'_k, t'_k, p'_k$ as in the proof of Lemma 12.3 and denote $\overline{\chi}_k = s'_k t'_k$, which is an edge or a vertex in some 1–skeleton geodesic $s_k t_k$. Denote $\chi_k = \mathcal{S}^{-1}(\overline{\chi}_k)$. Observe that $\chi_k$ does not depend on the choice of $s_k, t_k, s'_k, t'_k, p'_k$, since it is determined by the distances $|s_k t_k|, |s_k p_k|, |t_k p_k|$. Lemmas 12.4 — 12.8 are devoted to studying the position of $\chi_k$ w.r.t. $\rho_k$ (the simplices of the Euclidean diagonal).

We refer to the path $(v_k)$ as one *boundary component* of $\Delta$, and to the path $(w_k)$ as the other boundary component.

Finally, note that in the lemma below we actually do not have to assume that $(\sigma_k), (\tau_k)$ are directed geodesics.

**Lemma 12.4.** *In the above setting, assume that for all $i \leq k \leq j$ we have $p_k \neq p'_k$ (this does not depend on the choice of $p'_k$). Then for $i \leq k < j$,*

*(i) if $\chi_k, \chi_{k+1}$ are both edges, then they both intersect the same boundary component,*

*(ii) if one of $\chi_k, \chi_{k+1}$, say $\chi_k$, is an edge, and the second is a vertex, then either $\chi_k, \chi_{k+1}$ span a simplex, or they intersect the same boundary component,*

*(iii) if $\chi_k, \chi_{k+1}$ are both vertices, then they both lie on the same boundary component.*

*If we remove the assumption that $p_k \neq p'_k$, then in case (i) we only have that $\chi_k \subset S_1(\chi_{k+1})$ and $\chi_{k+1} \subset S_1(\chi_k)$, case (ii) remains unchanged, and in case (iii) we only have that $\chi_k, \chi_{k+1}$ span an edge.*

**Proof.** First let us prove the last assertion. We need to prove (up to interchanging $k$ with $k+1$) that for a vertex $u_0 \in \chi_k$ either there exists a neighbor of $u_0$ in $\chi_{k+1}$, or $\chi_k, \chi_{k+1}$ intersect the same boundary component. Suppose the first part of this alternative does not hold. Then, up to interchanging $v_k$ with $w_k$, we have the following configuration (which it will take some time to describe, since we need to name all the vertices that come into play):

We have $u_0 \neq w_k$, and we denote by $u_1$ the vertex following $u_0$ on 1–skeleton geodesic in $\Delta$ from $u_0$ to $w_k$, and by $u_2$ the vertex following $u_1$ if $u_1 \neq w_k$. In the layer $k+1$ we denote by $z_1 \neq w_{k+1}$ the vertex in the residue of $u_0u_1$ and by $z_2$ the vertex following $z_1$ on 1–skeleton geodesic $z_1w_{k+1}$. The configuration is the following: $\chi_{k+1}$ lies on the 1–skeleton geodesic $z_2w_{k+1}$.

Fix some 1–skeleton geodesics $s_l \dots s'_l, t_l \dots t'_l, p_l \dots p'_l$ for $l = k, k+1$. Consider a partial characteristic surface $S\colon \Delta \to X$ such that for $l = k, k+1$ we have that $S(v_lw_l)$ (where $v_lw_l$ is the 1–skeleton geodesic in $\Delta$) contains $s_l \dots s'_l$ and $t'_l \dots t_l$ (this is possible by Proposition 7.6). Then $S(z_2) \in s_{k+1} \dots s'_{k+1} \subset s_{k+1} \dots s'_{k+1}p'_{k+1} \dots p_{k+1}$ (where possibly $s'_{k+1} = p'_{k+1}$). By Proposition 7.6 applied to the partial characteristic surface for $p_k, p_{k+1}, s_k, s_{k+1}$, there is a neighbor of $S(z_2)$ on $s_k \dots s'_kp'_k \dots p_k$ (where possibly $s'_k = p'_k$). Denote this neighbor by $\overline{x}$. Since $S(u_0) \in \overline{\chi}_k$, we have that $\overline{x} \neq S(u_1)$, $\overline{x} \neq S(u_2)$. Moreover, since the vertices in the 1–skeleton geodesic $v_ku_0$ are not neighbors of $z_2$, we have by Lemma 11.4(iii) that $\overline{x} \notin s_k \dots s'_k$. So $\overline{x} \in p'_k \dots p_k$. But by Lemma 2.8 the vertices $\overline{x}, S(u_1)$, together with $S(u_2)$, if defined, span a simplex. On the other hand, $S(u_1)$, and $S(u_2)$ if defined, lie on the 1–skeleton geodesic $p_k \dots p'_kt'_k \dots t_k$ passing through $\overline{x}$. Since $\overline{x}, S(u_1)$, and $S(u_2)$, if defined, are different vertices, this is only possible if $\overline{x} = p'_k, S(u_0) = s'_k, S(u_1) = t'_k$ and $u_1 = w_k$, i.e. $u_2$ is not defined. Then $\chi_k$ is an edge, $\chi_{k+1}$ is a vertex, and they intersect the same boundary component, which is the second possibility of the alternative. Thus we have proved the last assertion of the lemma. In particular, we have proved assertion (ii).

Now we will be proving assertions (i,iii) and we may already assume that $p_k \neq p'_k$ for $i \leq k \leq j$.

First we prove (i), by contradiction. Suppose that $\chi_k, \chi_{k+1}$ are both edges, and w.l.o.g. suppose that $\chi_k$ does not intersect the boundary. This implies that $s'_k \neq s_k, t'_k \neq t_k$. Let $\overline{z}$ be a vertex in the projection (c.f. Definition 2.9) of the triangle $s'_kt'_kp'_k$ onto the layer $L_{k+1}$. By Lemma 10.5 applied thrice we get that $\overline{z}$ lies on 1–skeleton geodesics between all pairs of vertices from $\{s_{k+1}, t_{k+1}, p_{k+1}\}$, thus $\overline{\chi}_{k+1}$ is a vertex. Contradiction.

Now we prove (iii), by contradiction. Suppose that $\chi_k, \chi_{k+1}$ are both vertices and one of them is non–boundary. Then in the layers $k, k+1$ of $\Delta$ there are vertices, which are common neighbors of $\chi_k, \chi_{k+1}$, denote them by $u$ (in the layer $k$) and by $z$ (in the layer $k+1$). Moreover, either $\chi_k \neq v_k$ and $\chi_{k+1} \neq v_{k+1}$, or $\chi_k \neq w_k$ and $\chi_{k+1} \neq w_{k+1}$. Assume w.l.o.g that the latter holds. Consider the partial characteristic disc $\Delta_{pt}$ for $p_k, p_{k+1}, t_k, t_{k+1}$ (we are allowed to do this, since $|p_k t_k| = |p_k \overline{\chi}_k| + |\overline{\chi}_k t_k| \geq 2$ and similarly $|p_{k+1} t_{k+1}| \geq 2$) and the corresponding partial characteristic mapping $\mathcal{S}_{pt}$. Let $x$ be the common neighbor of $\mathcal{S}_{pt}^{-1}(\overline{\chi}_k), \mathcal{S}_{pt}^{-1}(\overline{\chi}_{k+1})$ in $\Delta_{pt}$ lying on $\mathcal{S}_{pt}^{-1}(p_k \overline{\chi}_k)$ or $\mathcal{S}_{pt}^{-1}(p_{k+1} \overline{\chi}_{k+1})$. Assume, w.l.o.g., that $\mathcal{S}_{pt}(x) \subset L_k$. Since vertices in $\mathcal{S}_{pt}(x), \mathcal{S}(u) \subset L_k$ are neighbors of $\overline{\chi}_{k+1} \in L_{k+1}$, we have by Lemma 2.8 that $\mathcal{S}_{pt}(x)$ and $\mathcal{S}(u)$ span a simplex. On the other hand, $\overline{\chi}_k$ lies by definition on some 1–skeleton geodesic $p_k s_k$. By Proposition 7.6, its segments $p_k \overline{\chi}_k$ and $\overline{\chi}_k s_k$ intersect $\mathcal{S}_{pt}(x)$ and $\mathcal{S}(u)$, respectively (outside $\overline{\chi}_k$). Hence $\overline{\chi}_k$ separates vertices from $\mathcal{S}_{pt}(x)$ and $\mathcal{S}(u)$ on a 1–skeleton geodesic $p_k s_k$. Contradiction. Thus we have proved assertion (iii) and hence the whole lemma. □

Let us introduce the following language.

**Definition 12.5.** We will refer to the *horizontal coordinates* of points in various characteristic discs. Namely, we view a characteristic disc as a $CAT(0)$ subspace of $\mathbb{E}^2$. There we consider cartesian coordinates such that the layers are contained in horizontal lines. We also specify that the horizontal coordinate increases (from the left to the right) in the direction from $v_k$ to $w_k$. We denote the horizontal coordinate of a point $z$ by $z^x$. If $\lambda$ is a vertical line in $\Delta$, then its horizontal coordinate is denoted by $\lambda^x$.

We will need the following technical lemma, which helps to compare the horizontal coordinates of the preimages of vertices of $X$ in various characteristic discs.

**Lemma 12.6.** *Suppose $\Delta^1, \Delta^2$ are partial characteristic discs (and $\mathcal{S}^1, \mathcal{S}^2$ resp. characteristic mappings) for the interval $(i, j)$ for some sequences of simplices $(\sigma_k^1), (\tau_k^1), (\sigma_k^2), (\tau_k^2)$ in the layers $L_k$ between $\sigma, \tau$. Suppose $(p_k)_{k=i}^j, (\tilde{p}_k)_{k=i}^j$ are 1–skeleton geodesics such that for $i \leq k \leq j$ we have that $p_k, \tilde{p}_k \in L_k$ and, for $l = 1, 2$, we have $p_k, \tilde{p}_k \in \mathcal{S}^l(\Delta^l)$ but $(\mathcal{S}^l)^{-1}(p_k) \neq (\mathcal{S}^l)^{-1}(\tilde{p}_k)$. Then, if we vary $i \leq k \leq j$, the differences between $((\mathcal{S}^1)^{-1}(p_k))^x$ and between $((\mathcal{S}^2)^{-1}(p_k))^x$ agree.*

**Proof.** Apply Lemma 11.4(iii). □

The following notions will help us formulate neatly the upcoming lemma.

**Definition 12.7.** Let $\Delta$ be a simplicial generalized characteristic disc for $(i, j)$ such that $|v_k w_k| \geq 2$ for $i \leq k \leq j$. Let $\chi, \rho$ be some simplices in the layer $k$ of $\Delta$, and $c \in \mathbb{Z}_+$. We say that $\chi$ is

*$\partial$–left* if either $v_k \in \chi$ or $\chi$ is a neighbor vertex of $v_k$, which has defect 1 in case $k \neq i, j$ or defect 2 in case $k = i$ or $k = j$,

*$\partial$–right* if either $w_k \in \chi$ or $\chi$ is a neighbor vertex of $w_k$, which has defect 1 in case $k \neq i, j$ or defect 2 in case $k = i$ or $k = j$,

*$c$–left* from $\rho$ if $|\chi, \rho| \geq c$ and $\chi$ lies on $v_k \rho$,

*$c$–right* from $\rho$ if $|\chi, \rho| \geq c$ and $\chi$ lies on $\rho w_k$.

In all that follows, $c$ is a positive integer. When all the pieces of the proof of Proposition 12.1 are put together, we assign $c = 5$. But before this happens, we use the variable $c$, in order to help keeping track of the role of the constant in the various lemmas.

**Lemma 12.8.** *Assume that for some $i < j$ and each $i \leq k \leq j$ the layer $k$ is thick for $(\sigma_k), (\tau_k)$, and $|p_k, \delta_k| \geq c + 4$. Then there exist $i \leq l \leq m \leq j$ such that*

*(i) for $i \leq k < l$ we have that $\chi_k$ is $\partial$–right or $c$–right from $\rho_k$,*

*(ii) among $l \leq k \leq m$ the differences between $(\mathcal{S}_{pr}^{-1}(r_k))^x$ are $\leq c + 1$,*

*(iii) for $m < k \leq j$ we have that $\chi_k$ is $\partial$–left or $c$–left from $\rho_k$.*

*Moreover, if the maximal thickness of the layers (for $(\sigma_k), (\tau_k)$) from $i$ to $j$ is $\geq 2c + 4$ and the layers $i - 1, j + 1$ are thin, then there are $l, m$ as above such that either $m < j$ and $v_{j+1}^x - v_{m+1}^x \geq c$ (in the characteristic disc for the thick interval $(i - 1, j + 1)$) or $l > i$ and $w_{l-1}^x - w_{i-1}^x \geq c$.*

The ranges for $k$ in (i),(ii),(iii), define the "initial" subinterval, the "middle" subinterval and the "final" subinterval of a "thick" interval discussed in the outline of the proof of Proposition 12.1. The last assertion, in the language of the outline, states that a "very thick" interval has either its "initial" or "final" subinterval non–"thin".

**Proof.** First we give the proof of (i)–(iii) under an additional assumption that for all $i \leq k \leq j$ we have $p_k \neq p'_k$ (recall that this does not depend on the choice of $p'_k$). The outline of the proof with this assumption was already given at the beginning of the section.

To start, observe that from Lemma 12.4 and Lemma 9.16(i,ii) we get immediately the following.

**Corollary.** There exist $i \leq l' \leq m' \leq j$ such that
(1) for $i \leq k < l'$ the simplex $\chi_k$ is $\partial$–right,
(2) for $l' \leq k \leq m'$ the simplices $\chi_k$ are alternatingly edges and vertices and their barycenters lie on a straight vertical line $\lambda$ in $\Delta$; moreover for $l' < k < m'$ the simplices $\chi_k$ do not meet $v_k, w_k$,
(3) for $m' < k \leq j$ the simplex $\chi_k$ is $\partial$–left.

Recall that the restriction to $\Delta$ of the $CAT(0)$ diagonal $\gamma'$ (c.f. Definition 9.10) in the characteristic disc containing $\Delta$ crosses transversally each vertical line in $\Delta$, by Lemma 9.17 (since $(j+1)-(i-1) > 2$). Let $l' \leq l \leq m'$ be maximal satisfying $(\gamma' \cap v_k w_k)^x \leq \lambda^x - c - \frac{1}{2}$ for $l' \leq k < l$. Similarly, let $l' \leq m \leq m'$ be minimal satisfying $(\gamma' \cap v_k w_k)^x \geq \lambda^x + c + \frac{1}{2}$ for $m < k \leq m'$.

We prove that assertion (i) is satisfied with $l$ as above. First consider $i \leq k < l'$. Then assertion (i) follows from assertion (1) of the corollary. Now suppose that $l' \leq k < l$. Then, by definitions of $l$ and $\rho_k$, if $\rho_k$ is a vertex, then $\rho_k^x \leq \lambda^x - c - \frac{1}{2}$, and if $\rho_k$ is an edge then the horizontal coordinates of its vertices are $\leq \lambda^x - c$. Moreover, in case the latter inequality is an equality, we have that $\chi_k$ is a vertex. In all cases $\chi_k$ lies to the right of $\rho_k$ and the distance between them is $\geq c$, as desired. Analogically, assertion (iii) holds with $m$ as above.

Now we prove assertion (ii). Consider $l \leq k \leq m$. If $l = m = l'$ or $l = m = m'$, then (ii) follows immediately. Otherwise, by definition of $m, l$ we have $(\gamma' \cap v_l w_l)^x > \lambda^x - c - \frac{1}{2}$ and $(\gamma' \cap v_m w_m)^x < \lambda^x + c + \frac{1}{2}$, hence $\lambda^x - c - \frac{1}{2} < (\gamma' \cap v_k w_k)^x < \lambda^x + c + \frac{1}{2}$. By definition of $\rho_k$, via similar considerations as in the previous paragraph, we have that $\mathrm{diam}(\rho_k \cup \chi_k) \leq c + 1$ and $|\rho_k, \chi_k| \leq c$. By the former inequality we have that $p'_k$ are at distance $\leq c+1$ from $r_k$. (Record the latter one, i.e. $|\rho_k, \chi_k| \leq c$, which we will need later in the proof.)

We would like to compute the differences between $(\mathcal{S}_{pr}^{-1}(p'_k))^x$, when we vary $l \leq k \leq m$. These differences are equal to the differences between $(\mathcal{S}_{ps}^{-1}(p'_k))^x$ in $\Delta_{ps}$, where $\mathcal{S}_{ps}$ (resp. $\Delta_{ps}$) is the partial characteristic mapping (resp. partial characteristic disc) for $(p_k)_{k=l}^m, (s_k)_{k=l}^m$. To see this, it is enough to apply Lemma 12.6 with $(p'_k), (p_k)$ in place of $(p_k), (\tilde{p}_k)$, where we use our additional assumption $p_k \neq p'_k$.

We claim that $(\mathcal{S}_{ps}^{-1}(p'_k))^x$ vary at most by $\frac{1}{2}$ for $l \leq k \leq m$. Indeed, by our additional assumption and assertion (2) of the corollary we have, for $l < k < m$, that $p_k \neq p'_k,\ s_k \neq s'_k,\ t_k \neq t'_k$. Thus we can apply Lemma 12.4 with $(s_k), (p_k), (t_k)$ in place of $(\sigma_k), (\tau_k), (p_k)$ to obtain, for $l \leq k \leq m$, that the barycenters of $\mathcal{S}_{ps}^{-1}(p'_k s'_k)$ lie on a common vertical line in $\Delta_{ps}$. This justifies the claim.

Thus $(\mathcal{S}_{pr}^{-1}(p'_k))^x$ vary at most by $\frac{1}{2}$, for $l \leq k \leq m$. Let $\mu$ be the greater among (at most two) values attained by $(\mathcal{S}_{pr}^{-1}(p'_k))^x$. By the previous estimates we have that $(\mathcal{S}_{pr}^{-1}(r_k))^x \leq \mu + c + 1$. On the other hand, we have $\mu \leq (\mathcal{S}_{pr}^{-1}(r_k))^x$. Hence we obtain that the differences between $(\mathcal{S}_{pr}^{-1}(r_k))^x$ are $\leq c+1$, as desired.

Now we must remove the additional assumption that for all $i \leq k \leq j$ we have $p_k \neq p'_k$. We have now only the last assertion of Lemma 12.4 at our disposal.

Let $i \leq i_1 \leq j_1 < i_2 \leq j_2 < \ldots < i_q \leq j_q \leq j$, where $j_h < i_{h+1} - 1$ for $1 \leq h < q$, be such that only for $i_h \leq k \leq j_h$ our additional assumption is satisfied. For all other $i \leq k \leq j$, in particular, for $k = i_h - 1, j_h + 1$ (where $1 \leq h \leq q$), except possibly for $i_1 - 1$ if it equals $i - 1$, and $j_q + 1$ if it equals $j + 1$, we have $|\chi_k, \rho_k| \geq |p_k, \delta_k| - 1 \geq c + 3$. Thus for $k = i_h, j_h$, except possibly for $i_1$ if it equals $i$ and for $j_q$ if it equals $j$, we have, by Lemma 9.11 and by the last assertion of Lemma 12.4, that $|\chi_k, \rho_k| \geq c + 1$. So for all $k$ not contained in the (open) intervals $(i_h, j_h)$ we have $|\chi_k, \rho_k| \geq c + 1$.

Put for a moment $j_0 = i,\ i_{q+1} = j$. By the previous paragraph, by Lemma 9.11 and by the last assertion of Lemma 12.4, for any $0 \leq h \leq q$ and all $j_h \leq k \leq i_{h+1}$, either $\rho_k$ lies always between $\chi_k$ and $v_k$, or $\rho_k$ lies always between $\chi_k$ and $w_k$.

Now let us analyze what happens for a fixed $1 \leq h \leq q$ for $i_h \leq k \leq j_h$. Apply our argument under the additional assumption $p_k = p'_k$ to $i = i_h,\ j = j_h$. Observe that if $|\chi_{i_h}, \rho_{i_h}| \geq c + 1$ (which holds unless possibly $h = 1$ and $i_1 = i$) and $\chi_{i_h}$ lies between $\rho_{i_h}$ and $v_{i_h}$, then we have that $l = m = i_h$ (otherwise we have recorded that $|\rho_k, \chi_k| \leq c$ for $l \leq k \leq m$). Similarly, if $|\chi_{j_h}, \rho_{j_h}| \geq c + 1$ (which holds unless possibly $h = q$ and $j_q = j$) and $\chi_{j_h}$ lies between $\rho_{j_h}$ and $w_{j_h}$, then $l = m = j_h$. In particular, those two situations cannot happen simultaneously, and if any of them happens, then either assertion (i) or assertion (iii) is valid for all $i_h \leq k \leq j_h$.

Summarizing, there can be at most one $h$ such that $l \neq j_h$ and $m \neq i_h$. If there is no such $h$, then either assertion (i) or assertion (iii) holds for all $i \leq k \leq j$ and we are done. If not, define $l, m$ as in the previous argument for $i = i_h, j = j_h$. They satisfy assertions (i,ii,iii), as required.

Finally, we prove the last assertion. Pick $\lambda, l, m$ as above. Let $\Delta$ be the characteristic disc for $(i-1, j+1)$ and let $\gamma'$ be its $CAT(0)$ diagonal. Since the maximal thickness for $(\sigma_k), (\tau_k)$ of the layers from $i$ to $j$ is $\geq 2c+4$, then by Lemma 9.16(i,ii), we have that $v_{j+1} - w_{i-1} \geq 2c+1$. Thus we can assume w.l.o.g. that $\lambda^x - w_{i-1}^x \geq c + \frac{1}{2}$. Thus $\lambda^x - (\gamma' \cap v_i w_i)^x \geq c + \frac{1}{2}$ and $l > i$. Observe that $\lambda$ goes through the barycenter of $\chi_l$, hence $w_{l-1}^x \geq \lambda^x - \frac{1}{2}$ so $w_{l-1}^x - w_{i-1}^x \geq c$, as desired. $\square$

The next lemma in particular guarantees that in a "thick" interval, the vertices $\mathcal{S}_{pr}^{-1}(r_k)$ for $k$ in the "final" subinterval outside the "post–final" subinterval form a coarse vertical line. We consider it, together with the previous lemma, the heart of the proof of Proposition 12.1. Below we put $\Delta$ to be the characteristic disc for the thick interval containing $i, \ldots, j$ for $(\sigma_k), (\tau_k)$. Let $v_k, w_k$ be its boundary vertices, etc.

**Lemma 12.9.** *Suppose that for some $i \leq j$ and for all $i \leq k \leq j$ the layer $k$ is thick for $(\sigma_k), (\tau_k)$, $|p_k, \delta_k| \geq c+2 \geq 7$ and $\chi_k$ is either $\partial$–left or $c$–left from $\rho_k$. If $(\gamma' \cap v_{j+1}w_{j+1})^x = v_{j+1}^x + \frac{1}{2}$, then $v_{j+1}^x - v_i^x < c$.*

**Proof.** By contradiction. Roughly, the idea is the following. If $v_{j+1}^x$ is relatively large w.r.t. $v_i^x$, this means that the directed geodesic $(\sigma_k)$ performs in the layers $i, \ldots, j$ an unexpected turn towards $(\tau_k)$. On the other hand, there is plenty of room in the partial characteristic disc $\Delta_{pt}$ for $(p_k), (\tau_k)$, since $p_k$ are far away from $\delta_k$, hence away from $\sigma_k$. By assumption on $\chi_k$ the corresponding characteristic image $\mathcal{S}_{pt}(\Delta_{pt})$ almost passes through $\sigma_k$. We can then see through $\Delta_{pt}$ that $(\sigma_k)$ actually goes vertically for all consecutive $i \leq k \leq j$. This yields a contradiction.

Formally, suppose $v_{j+1}^x - v_i^x \geq c$. By increasing $i$, if necessary, we may assume that $i$ is maximal $\leq j$ satisfying $v_{j+1}^x - v_i^x \geq c$. Hence $v_{j+1}^x - v_i^x = c$.

We claim that for all $i \leq k \leq j$ we have that $\chi_k$ is $\partial$–left. Indeed, by maximality of $i$ we have $(\gamma' \cap v_{j+1}w_{j+1})^x - v_k^x \leq c + \frac{1}{2}$. By Lemma 9.17 we have that $(\gamma' \cap v_k w_k)^x - (\gamma' \cap v_{j+1}w_{j+1})^x < 0$. Putting these inequalities together implies that $|v_k, \rho_k| \leq c$. Hence if $\chi_k$ is $c$–left from $\rho_k$, then it equals $v_k$, thus it is also $\partial$–left, as required. Thus we have proved the claim. Moreover, $|v_k, \rho_k| \leq c$ together with $|p_k, \delta_k| \geq c+2$ gives also that $|p_k, \sigma_k| \geq 2$ and $p_k \neq t'_k$ for $i \leq k \leq j$.

Denote $h_k = \mathcal{S}^{-1}(t'_k) \in \chi_k$. By the claim we have $|v_k h_k| \leq 1$. Let $\Delta_{pt}$ be the characteristic disc for the thick interval $(i_{pt}, j_{pt})$ for $(p_k), (t_k)$ containing $i \leq k \leq j$ and let $\mathcal{S}_{pt}$ be the corresponding characteristic mapping (we have $|p_k t_k| = |p_k t'_k| + |t'_k t_k| \geq 2$, since $\chi_k$ is $\partial$–left). Denote $\tilde{v}_k = \mathcal{S}_{pt}^{-1}(p_k), \tilde{w}_k = \mathcal{S}_{pt}^{-1}(t_k)$. Let $\tilde{h}_k = \mathcal{S}_{pt}^{-1}(t'_k)$. Since for $i \leq k \leq j$ we have $|t_k t'_k| \geq 1$, by Lemma 12.6 the differences between $h_k^x$ (coordinates in $\Delta$) and $\tilde{h}_k^x$ (coordinates in $\Delta_{pt}$) agree.

Now observe that $t'_k$ spans a simplex with $\sigma_k$ by the claim, Lemma 10.3 and Lemma 9.8(iii,iv), for all $i \leq k \leq j$. Denote $\phi = \mathrm{span}\{t'_i, \sigma_i\}$. Denote by $\phi_i = \phi, \phi_{i+1}, \ldots$ the simplices of the directed geodesic from $\phi$ to $\tau$. Denote by $\beta_k$ the simplices of the directed geodesic from $t'_i$ to $\tau$. By Lemma 2.10 we have $\beta_k \subset \phi_k \supset \sigma_k$ for $k-i$ even, and $\beta_k \supset \phi_k \subset \sigma_k$ for $k-i$ odd. Denote by $\alpha_k$ the simplices of the directed geodesic in $\Delta_{pt}$ from $\tilde{h}_i$ to $\tilde{v}_{j_{pt}}\tilde{w}_{j_{pt}}$.

First we prove that for all $i \leq k \leq j$ we have $\tilde{v}_k \notin \alpha_k$. For $k = i$ this follows from $p_i \neq t'_i$. For $k > i$ we argue by contradiction. Let $i < k_0 \leq j$ be minimal such that $\tilde{v}_{k_0} \in \alpha_{k_0}$. Observe that $\Delta_{pt}$ is actually a partial characteristic disc for $(p_k), (\tau_k)$ and $(\tau_k)$ is the directed geodesic from $\tau$ to $\sigma$. Hence, similarly as in Lemma 10.6, for $i \leq k \leq k_0$ the simplices $\alpha_k$ are alternatingly vertices and edges, with barycenters on a common vertical line. Moreover, by minimality of $k_0$, we have that $\alpha_{k_0}$ is an edge. By Lemma 10.5 and Lemma 2.10 (applied alternatingly for consecutive layers exactly as in the proof of Proposition 10.2), we have that $\beta_k \subset \mathcal{S}_{pt}(\alpha_k)$ for $k - i$ even and $\mathcal{S}_{pt}(\alpha_k) \subset \beta_k$ for $k - i$ odd, for all $i \leq k \leq k_0$. In particular, since $\alpha_i$ is a vertex and $\alpha_{k_0}$ is an edge, we have that $p_{k_0} \in \mathcal{S}_{pt}(\alpha_{k_0}) \subset \beta_{k_0} \supset \phi_{k_0} \subset \sigma_{k_0}$. But this contradicts $|p_{k_0}, \sigma_{k_0}| \geq 2$. Hence we proved that for all $i \leq k \leq j$ we have $\tilde{v}_k \notin \alpha_k$.

From the above proof we also get that for all $i \leq k \leq j$ we have $\beta_k \subset \mathcal{S}_{pt}(\alpha_k)$ for $k-i$ even and $\mathcal{S}_{pt}(\alpha_k) \subset \beta_k$ for $k-i$ odd, and the simplices $\alpha_k$ are alternatingly vertices and edges, with barycenters on a common vertical line. Since $t'_k$ and $\sigma_k$ span a simplex, this implies that $t'_k \in B_2(\mathcal{S}_{pt}(\alpha_k))$, hence $\tilde{h}_k \in B_2(\alpha_k)$, for $i \leq k \leq j$. Since the barycenters of $\alpha_k$ lie on a common vertical line through $\tilde{h}_i$, we conclude that $|\tilde{h}^x_i - \tilde{h}^x_k| \leq 2\frac{1}{2}$ for $i \leq k \leq j$, in particular for $k = j$. But $\tilde{h}^x_j - \tilde{h}^x_i = h^x_j - h^x_i \geq c - 1\frac{1}{2}$. This contradicts $c \geq 5$. □

We immediately get the following corollary, which excludes the possibility of adjoining a non–"thin" "final" subinterval of a "thick" interval to the beginning of a thin layer for $(\sigma_k), (\tau_k)$.

**Corollary 12.10.** *Suppose that for some $i \leq j$ the layer $j + 1$ is thin for $(\sigma_k), (\tau_k)$, and for all $i \leq k \leq j$ the layer $k$ is thick for $(\sigma_k), (\tau_k)$, $|p_k, \delta_k| \geq c + 2 \geq 7$ and $\chi_k$ is either $\partial$–left or $c$–left from $\rho_k$. Then $v^x_{j+1} - v^x_i < c$.*

The next preparatory lemma takes care of the "thin" intervals for $(\sigma_k), (\tau_k)$. Let $d$ be a positive integer.

**Lemma 12.11.** *Suppose that for some $i \leq j$ the layers $i, j$ for $(\sigma_k), (\tau_k)$ have thickness $\leq d$ and for all $i \leq k \leq j$ the layer $k$ for $(\sigma_k), (\tau_k)$ has thickness $\leq 2c + 3$ and $|p_k, \delta_k| \geq 2c + 4$. Then the differences between $(\mathcal{S}^{-1}_{pr}(r_k))^x$ are $\leq c + 2d + 3\frac{1}{2}$.*

We can also obtain an estimate independent of $c$ on the differences between $(\mathcal{S}^{-1}_{pr}(r_k))^x$. However, we will not need it.

**Proof.** We can define $p'_k$ as usual (even for thin layers). Observe that we have $p_k \neq p'_k$, $|p_k, \sigma_k| \geq 2$, and $|p_k, \tau_k| \geq 2$, for $i \leq k \leq j$. Let $\tilde{s}_k \in \sigma_k, \tilde{t}_k \in \tau_k$

realize maximal distances from $p_k$ to $\sigma_k, \tau_k$, respectively. Let $\Delta_{ps}, \Delta_{pt}, \mathcal{S}_{ps}, \mathcal{S}_{pt}$ denote the characteristic discs and mappings for $(p_k), (\sigma_k)$ and $(p_k), (\tau_k)$, respectively, for the thick intervals containing all $i \leq k \leq j$. Since $p_k \neq p'_k$, we have by Lemma 12.6 that the differences between $(\mathcal{S}_{ps}^{-1}(p_k))^x$, between $(\mathcal{S}_{pt}^{-1}(p_k))^x$, and between $(\mathcal{S}_{pr}^{-1}(p_k))^x$ agree, if we vary $k$ among $i \leq k \leq j$.

For $i \leq k \leq j$ denote $\dot{s}_k = \mathcal{S}_{ps}^{-1}(\tilde{s}_k), \dot{t}_k = \mathcal{S}_{pt}^{-1}(\tilde{t}_k)$. Let $i \leq k_1 < k_2 \leq j$. By Lemma 9.16(i,ii) we have that $\dot{s}_{k_1}^x - \dot{s}_{k_2}^x \geq -\frac{1}{2}$ and $\dot{t}_{k_1}^x - \dot{t}_{k_2}^x \leq \frac{1}{2}$. In particular, $\dot{s}_{k_2}^x - \dot{s}_j^x \geq -\frac{1}{2}$ and $\dot{s}_i^x - \dot{s}_{k_1}^x \geq -\frac{1}{2}$, for $i \leq k_1 < k_2 \leq j$. Hence

$$\dot{s}_{k_2}^x - \dot{s}_{k_1}^x \geq \dot{s}_j^x - \dot{s}_i^x - 1 \geq \dot{t}_j^x - \dot{t}_i^x - 1 - 2d \geq -2d - 1\frac{1}{2}.$$

Analogically,

$$\dot{t}_{k_2}^x - \dot{t}_{k_1}^x \leq 2d + 1\frac{1}{2}.$$

It will be convenient for us to assume that the coordinates in $\Delta_{ps}, \Delta_{pt}$ agree on $\mathcal{S}_{ps}^{-1}(p_k)$ and $\mathcal{S}_{pt}^{-1}(p_k)$, so that we can compare coordinates of points in $\Delta_{ps}$ and $\Delta_{pt}$. With this convention, for any $i \leq k_1, k_2 \leq j$ we have that $\dot{s}_{k_1}^x - \dot{t}_{k_2}^x \geq \dot{s}_j^x - \dot{t}_j^x - 1 \geq -d-1$. Analogically $\dot{s}_{k_1}^x - \dot{t}_{k_2}^x \leq d+1$. So altogether the differences between all the $\dot{s}_k^x, \dot{t}_k^x$, where $i \leq k \leq j$, are $\leq 2d + 1\frac{1}{2}$. In particular, if we denote by $a$ the minimum over $k$ of $\dot{s}_k^x, \dot{t}_k^x$ and by $b$ the maximum over $k$ of $\dot{s}_k^x, \dot{t}_k^x$, we get $b - a \leq 2d + 1\frac{1}{2}$.

For a fixed $k$, since the thickness of the layer $k$ is $\leq 2c+3$, we have that $|\tilde{s}_k r_k| \leq c+1$ or $|\tilde{t}_k r_k| \leq c+1$, hence

$$\min\{|p_k\tilde{s}_k|, |p_k\tilde{t}_k|\} \leq |p_k r_k| + c + 1,$$

thus $r_k^x \geq a - (c+1)$. On the other hand, we have

$$|p_k r_k| \leq \max\{|p_k\tilde{s}_k|, |p_k\tilde{t}_k|\} + 1,$$

hence $r_k^x \leq b + 1$. This altogether implies that the differences between $(\mathcal{S}_{pr}^{-1}(r_k))^x$ are

$$\leq (c+1) + \left(2d + 1\frac{1}{2}\right) + 1 = c + 2d + 3\frac{1}{2}.$$

□

Finally, we prove the following easy lemma, which will be needed also later in Section 13.

**Lemma 12.12.** *Let $\Delta$ be a generalized characteristic disc for $(i,j)$. Let $\gamma$ be a $CAT(0)$ geodesic in $\Delta$ connecting some points in $v_i w_i, v_j w_j$. For $i \leq k \leq j$*

*let $h_k \in v_k w_k$ be some points at distance $\le \frac{1}{2}$ from $\gamma \cap v_k w_k$. Let $\Delta_{split} \subset \Delta$ be the generalized characteristic disc for $(i, j)$ with $w_k$ substituted with $h_k$. Then the $CAT(0)$ geodesic $h_i h_j$ in $\Delta_{split}$ is 1–close to the piecewise linear boundary path $h_i h_{i+1} \dots h_j$.*

**Proof.** For $i \le k \le j$ let $h'_k$ be the points on $v_k w_k$ with $(h'_k)^x = \max\{(\gamma \cap v_k w_k)^x, h_k^x\}$. Let $\Delta_{cut} \subset \Delta$ be the generalized characteristic disc for $(i, j)$ with $w_k$ substituted with $h'_k$. Then $\gamma$ is also a $CAT(0)$ geodesic in $\Delta_{cut}$. By Lemma 11.7 applied to $\Delta_{split} \subset \Delta_{cut}$ we have that the $CAT(0)$ geodesic $h_i h_j$ in $\Delta_{split}$ is $\frac{1}{2}$–close to $\gamma$, hence 1–close to the path $h_i h_{i+1} \dots h_j$. □

Now we are ready to put together all pieces of the puzzle.

**Proof of Proposition 12.1.** Put $c = 5$. For the layers $k$ such that $|p_k r_k| \le 7c + 13$ there is nothing to prove. Now suppose that for some $i' < j'$, where $j' - i' \ge 2$, we have $|p_{i'} r_{i'}| = |p_{j'} r_{j'}| = 7c + 13$ and for $i' < k < j'$ we have $|p_k r_k| \ge 7c + 14$, hence $|p_k, \delta_k| \ge 7c + 13$. In particular, $p_k$ are as far from $\delta_k$ as required in Lemma 12.9 and Corollary 12.10.

Let $\Delta_{pr}$ be the partial characteristic disc for $(i', j')$ for $(p_k), (r_k)$, and let $\mathcal{S}_{pr}$ be the corresponding partial characteristic mapping. Denote $u_k = \mathcal{S}_{pr}^{-1}(r_k)$.

**Step 1.** There exist $i' \le l \le m \le j'$ such that
(1) for $i' \le k < l$ the layer $k$ is thick for $(\sigma_t), (\tau_t)$, some 1–skeleton geodesic $p_k s_k$ intersects $\delta_k$, and $(\gamma' \cap v_k w_k)^x < w_k^x - \frac{1}{2}$ (in the appropriate characteristic disc for $(\sigma_t), (\tau_t)$, with the usual notation $v_k, w_k$, etc.),
(2) among $l \le k \le m$ the differences between $u_k^x$ are $\le 7c + 11\frac{1}{2}$,
(3) for $j' \ge k > m$ the layer $k$ is thick for $(\sigma_t), (\tau_t)$, some 1–skeleton geodesic $p_k t_k$ intersects $\delta_k$, and $(\gamma' \cap v_k w_k)^x > v_k^x + \frac{1}{2}$.

This is the division into the "pre–initial" interval, the union of the central intervals, and the "post–final" interval in the language of the outline of the proof.

Let us justify Step 1. First consider the simple case that there are no thin layers for $(\sigma_k), (\tau_k)$ among the layers $i' \le k \le j'$. Then Lemma 12.8 applied to $i = i'$, $j = j'$ gives us a pair of numbers $l', m'$, which satisfies assertions (1) and (3) of Step 1 (with $l', m'$ in place of $l, m$), except for the statements on the position of $\gamma'$ (we will refer to these as *incomplete* assertions (1),(3)).

Let $l < l'$ be minimal $\ge i'$ such that $(\gamma' \cap v_l w_l)^x = w_l^x - \frac{1}{2}$ (if there is no such $l$, in particular, if $l' = i'$, then we put $l = l'$). Similarly, let $m > m'$ be maximal $\le j'$ such that $(\gamma' \cap v_m w_m)^x = v_m^x + \frac{1}{2}$ (if there is no such $m$, in particular, if $m' = j'$, then we put $m = m'$). Obviously, $l, m$ satisfy complete assertions (1) and (3) of Step 1. To prove that they satisfy assertion (2), we need the following.

**Claim.** Among $l \leq k \leq l' - 1$ the differences between $u_k^x$ are $\leq c+1$. Analogically, among $m'+1 \leq k \leq m$ the differences between $u_k^x$ are $\leq c+1$.

To justify the claim, we need to introduce some notation. Up to the end of the proof of the claim we consider $l \leq k \leq l'-1$. Observe that the layers $k$ for $(p_k), (s_k)$ are thick, since by incomplete assertion (1) we have that $|p_k s_k| > |p_k, \delta_k|$. Denote by $\Delta, \mathcal{S}$ (resp. $\Delta_{ps}, \mathcal{S}_{ps}$) the characteristic disc and mapping for the thick interval containing $k$ for $(\sigma_t), (\tau_t)$ (resp. for $(p_t), (s_t)$). For each $k$ choose a vertex $\overline{h}_k$ in $\delta_k \cap p_k s_k$ closest to $p_k$. By Proposition 7.6 we have that $\overline{h}_k \in \mathcal{S}_{ps}(\Delta_{ps})$. Denote $h_k = \mathcal{S}^{-1}(\overline{h}_k)$, $\tilde{h}_k = \mathcal{S}_{ps}^{-1}(\overline{h}_k)$. Since by incomplete assertion (1) we have $s_k \neq s'_k$, Lemma 12.6 gives that the differences between $-h_k^x$ and between $\tilde{h}_k^x$ agree (the sign changes since $(s_k)$ plays the role of the left boundary component in $\mathcal{S}(\Delta)$ and the right one in $\mathcal{S}_{ps}(\Delta_{ps})$). By Lemma 12.6 applied to $\Delta_{ps}$ and $\Delta_{pr}$, and since $|p_k r_k| = |p_k \overline{h}_k|$ or $|p_k r_k| = |p_k \overline{h}_k| + 1$, we have that the differences between $u_k^x$ differ at most by 1 from the differences between $\tilde{h}_k^x$. Hence the differences between $u_k^x$ differ at most by 1 from the differences between $-h_k^x$.

Now we can proceed with justifying the claim. By Lemma 12.9 we have that $w_{l'-1}^x - w_l^x < c$, hence $(\gamma' \cap v_l w_l)^x \geq w_{l'-1}^x - c$. Thus, by Lemma 9.17, we have $(\gamma' \cap v_k w_k)^x \geq w_{l'-1}^x - c$ for all $k$. This implies, by the definition of $\rho_k$, that $h_k^x \geq w_{l'-1}^x - c - \frac{1}{2}$. On the other hand, by Lemma 9.16 we have that $w_k^x \leq w_{l'-1}^x + \frac{1}{2}$, hence we have $h_k^x \leq w_{l'-1}^x - \frac{1}{2}$. Thus the differences between $h_k^x$ are $\leq c$, hence the differences between $u_k^x$ are $\leq c+1$. This justifies the first assertion of the claim. The second one is proved analogically.

Now we can finish the proof of Step 1 in the simple case that there are no thin layers for $(\sigma_k), (\tau_k)$, among the layers $i' \leq k \leq j'$. To prove assertion (2), we need to compare $u_{k_1}^x$ and $u_{k_2}^x$, for $l \leq k_1 < k_2 \leq m$. Assume, which is the worst possible case, that $l \leq k_1 \leq l'-1$ and $m'+1 \leq k_2 \leq m$. By Lemma 12.8(ii) and by the claim we have

$$|u_{k_1}^x - u_{k_2}^x| \leq |u_{k_1}^x - u_{l'-1}^x| + \frac{1}{2} + |u_{l'}^x - u_{m'}^x| + \frac{1}{2} + |u_{m'+1}^x - u_{k_2}^x| \leq$$
$$\leq (c+1) + \frac{1}{2} + (c+1) + \frac{1}{2} + (c+1),$$

which is even better then the required estimate. This ends the proof of Step 1 in the simple case.

Now consider the complex case that that there is a thin layer among the layers $i' \leq k \leq j'$. Let $(l_0, m_0)$ be a maximal (w.r.t. inclusion) interval, with

$i' \le l_0 \le m_0 \le j'$, such that the layers $l_0, m_0$ are thin for $(\sigma_k), (\tau_k)$ and for $l_0 < k < m_0$ the layer $k$ has thickness $\le 2c+3$ (possibly $l_0 = m_0$). This is the "proper thin" interval of the outline of the proof.

First we argue that for $i' \le k < l_0$ and $m_0 < k \le j'$ the layer $k$ is thick. Otherwise, suppose w.l.o.g. that $k_0$ is maximal $< l_0$ such that the layer $k_0$ is thin. Then, by maximality of $(l_0, m_0)$, the thick interval $(k_0, l_0)$ contains some $k$ such that the layer $k$ has thickness $\ge 2c+4$. Thus by the last assertion of Lemma 12.8 applied to $i = k_0+1, j = l_0 - 1$ we get $k_0 < l \le m < l_0$ so that either $m < l_0 - 1$ and $v^x_{l_0} - v^x_{m+1} \ge c$, or $l > k_0 + 1$ and $w^x_{l-1} - w^x_{k_0} \ge c$. In both cases this contradicts Corollary 12.10 applied respectively to $i = m+1, j = l_0 - 1$, or to $i = l-1, j = k_0+1$ with the roles of $v, w$ interchanged and the order on naturals inversed. Thus we have proved that for $i' \le k < l_0$ and $m_0 < k \le j'$ the layer $k$ is thick for $(\sigma_k), (\tau_k)$.

Now we can apply Lemma 12.8 to $i = i',\ j = l_0 - 1$. Denote by $l', m'$ the pair of numbers given by its assertion. By Corollary 12.10 we have that $v^x_{l_0} - v^x_k < c$ for $m'+1 \le k \le l_0$. Similarly, we apply Lemma 12.8 to $i = m_0 + 1,\ j = j'$ and denote by $l'', m''$ the pair of numbers given by its assertion. By Corollary 12.10 we have $w^x_k - w^x_{m_0} < c$ for $m_0 \le k \le l'' - 1$. Hence, by Lemma 9.16(i,ii), the thickness of the layer $k$, for $m'+1 \le k \le l_0$ and for $m_0 \le k \le l' - 1$, is $\le c+1$.

Define, similarly as before, $l < l'$ to be minimal $\ge i'$ such that $(\gamma' \cap v_l w_l)^x = w^x_l - \frac{1}{2}$ (if there is no such $l$, in particular, if $l' = i'$, then we put $l = l'$), in appropriate characteristic disc. Similarly, let $m > m''$ be maximal $\le j'$ such that $(\gamma' \cap v_m w_m)^x = v^x_m + \frac{1}{2}$ (if there is no such $m$, in particular, if $m'' = j'$, then we put $m = m''$).

For $l, m$ as above we have that assertion (1) follows from Lemma 12.8(i) and assertion (3) follows from Lemma 12.8(iii). As for assertion (2), assume, which is the worst possible case, that $l \le k_1 \le l' - 1$ and $m''+1 \le k_2 \le m'$. Combining Lemma 12.11 applied to $i = m'+1,\ j = l''-1,\ d = c+1$ with Lemma 12.8(ii) and with the claim above (which is also valid in this complex case) we get

$$\begin{aligned}|u^x_{k_1} - u^x_{k_2}| &\le |u^x_{k_1} - u^x_{l'-1}| + \frac{1}{2} + |u^x_{l'} - u^x_{m'}| + \frac{1}{2} + |u^x_{m'+1} - u^x_{l''-1}| + \\ &+ \frac{1}{2} + |u^x_{l''} - u^x_{m''}| + \frac{1}{2} + |u^x_{m''+1} - u^x_{k_2}| \le \\ &\le (c+1) + \frac{1}{2} + (c+1) + \frac{1}{2} + \Big(c + 2d + 3\frac{1}{2}\Big) + \\ &+ \frac{1}{2} + (c+1) + \frac{1}{2} + (c+1) = 7c + 11\frac{1}{2},\end{aligned}$$

as required. Thus we have completed the proof of Step 1.

**Step 2.** $\gamma_{pr}$ is 99–close to $(u_k)$.

For the layers $i' \leq k < l$ define $\Delta, \mathcal{S}, \Delta_{ps}, \mathcal{S}_{ps}$ and $h_k \in \rho_k \subset \Delta$, $\tilde{h}_k \in \Delta_{ps}$, $\overline{h}_k = \mathcal{S}(h_k) = \mathcal{S}_{ps}(\tilde{h}_k)$ like in Step 1 (which is possible by assertion (1) of Step 1). Recall that the differences between $u_k^x$ differ at most by 1 from the differences between $-h_k^x$. In particular, since for $i' \leq k_1 < k_2 < l$ we have $h_{k_1}^x - h_{k_2}^x \leq \frac{1}{2}$ (by Lemma 9.17 and the definition of $\rho_k$), it follows that $u_{k_2}^x - u_{k_1}^x \leq 1\frac{1}{2}$. Analogically we choose vertices $h_k \in \rho_k$ (in appropriate characteristic disc) for $m < k \leq j'$, so that $|p_k r_k| = |p_k \overline{h}_k|$ or $|p_k r_k| = |p_k \overline{h}_k| + 1$. Hence for $m < k_2 < k_1 \leq j'$ we have $u_{k_2}^x - u_{k_1}^x \leq 1\frac{1}{2}$.

Let $l \leq k_0 \leq m$ be such that $u_{k_0}^x$ is minimal. Let $\alpha$ be a vertical line segment in $\Delta_{pr}$ from the layer $\max\{l-1, i'+1\}$ to the layer $\min\{m+1, j'-1\}$ at distance 2 to the left from $u_{k_0}$. By assertion (2) of Step 1 and by the fact that $|p_k r_k| \geq 7c + 14$ this line segment is really contained in $\Delta_{pr}$. Let $\beta_1, \beta_2$ be $CAT(0)$ geodesics in $\Delta_{pr}$ connecting $u_{i'}, u_{j'}$ to the endpoints of $\alpha$. Since $u_{k_2}^x - u_{k_1}^x \leq 1\frac{1}{2}$ for $i' \leq k_1 < k_2 < l$ and $m < k_2 < k_1 \leq j'$, we have for all $i' \leq k \leq j'$ that $u_k^x > \alpha^x$. Hence the region in $\Delta_{pr}$ to the right of the concatenation $\beta_1 \alpha \beta_2^{-1}$ is convex, and thus contains the $CAT(0)$ geodesic in $\Delta_{pr}$ joining $u_{i'}$ with $u_{j'}$.

We claim that $\beta_1$ is $(7c+16)$–close to $(u_k)$. Indeed, if $l = i'$ or $l - 1 = i'$, then this is easy. Otherwise, let $i' \leq k \leq l-1$. Let $\Delta'' \subset \Delta|_{i'}^{l-1}$ be the generalized characteristic disc for $(i', l-1)$ obtained from $\Delta'|_{i'}^{l-1}$ (the modified characteristic disc, in which $\gamma'$ is a $CAT(0)$ geodesic) by substituting $w_k'$ with $w_k''$, such that $(w_k'')^x = h_k^x + 1$. Denote $\gamma'$ restricted to the layers from $i'$ to $l-1$ by $\gamma'|_{i'}^{l-1}$. We have $\gamma'|_{i'}^{l-1} \subset \Delta''$ and by assertion (1) of Step 1 we have that $\gamma'|_{i'}^{l-1}$ is in $\Delta''$ a $CAT(0)$ geodesic. Let $\Delta_{ps}' \subset \Delta_{ps}|_{i'}^{l-1}$ be the generalized characteristic disc for $(i', l-1)$ obtained from $\Delta_{ps}|_{i'}^{l-1}$ by deleting $\frac{1}{2}$–horizontal neighborhood of the boundary component corresponding to $(s_k)$. Observe that there is an (orientation reversing) embedding $e'' \colon \Delta'' \to \Delta_{ps}'$, and that $e''(\gamma')$ is still a $CAT(0)$ geodesic in $\Delta_{ps}'$. Moreover, $e''(h_k) = \tilde{h}_k$, so that $|e''(\gamma' \cap v_k w_k)\tilde{h}_k| \leq \frac{1}{2}$.

Let $\Delta_{ph} \subset \Delta_{ps}|_{i'}^{l-1}$ be the generalized characteristic disc for $(i', l-1)$ obtained from $\Delta_{ps}'$ by splitting along $\tilde{h}_k$ (in fact $\Delta_{ph}$ is the partial characteristic disc for $(p_k), (\overline{h}_k)$). By Lemma 12.12 the $CAT(0)$ geodesic $\tilde{h}_{i'}\tilde{h}_{l-1}$ in $\Delta_{ph}$ is 1–close to the boundary path $(\tilde{h}_k)$. Now recall that there is an embedding $e \colon \Delta_{ph} \to \Delta_{pr}$, such that $|e(\tilde{h}_k)u_k| \leq 1$. Let us compute the distances between the endpoints of the image under $e$ of the $CAT(0)$ geodesic $\tilde{h}_{i'}\tilde{h}_{l-1}$ and the endpoints of $\beta_1$ in $\Delta_{pr}$. The distance between $e(\tilde{h}_{i'})$ and $u_{i'}$ is $\leq 1$, and the distance between the second pair of endpoints is $\leq 2 + (7c + 11\frac{1}{2}) + \frac{1}{2}$ by assertion (2) of Step 1. Hence, by Lemma 11.7, we have that $e(\tilde{h}_{i'}\tilde{h}_{l-1})$

is $(7c+14)$–close to $\beta_1$. Recall that $e(\tilde{h}_{i'}\tilde{h}_{l-1})$ is 1–close to $e\big((\tilde{h}_k)\big)$, which is 1–close to $(u_k)$. Altogether, $\beta_1$ is $\big((7c+14)+1+1\big)$–close to $(u_k)$, as desired. Thus we have justified the claim. Analogically, $\beta_2$ is $(7c+16)$–close to $(u_k)$.

From the claim and since, by assertion (2) of Step 1, $\alpha$ is $(7c+14)$–close to $(u_k)$, it follows that the two boundary components of the convex region in $\Delta_{pr}$ to the right of $\beta_1\alpha\beta_2^{-1}$ are $(7c+16)$–close. Hence the $CAT(0)$ geodesic $u_{i'}u_{j'}$ in $\Delta_{pr}$ is $(7c+16)$–close to $(u_k)$. Now consider the $CAT(0)$ geodesic $\gamma_{pr}$ in $\Delta_{pr}$ (which appears in the statement of the proposition) restricted to the layers from $i'$ to $j'$. Since its endpoints are at distance $\leq 7c+13$ from the endpoints of $u_{i'}u_{j'}$ (this is because $|p_{i'}r_{i'}| = 7c+13 = |p_{j'}r_{j'}|$), we get (by Lemma 11.7) that $\gamma_{pr}$ is $(14c+29)$–close to $u_k$, as desired (recall that $c=5$). □

# 13 Contracting

In this section we prove the following consequence of Proposition 12.1, which summarizes the contracting properties of Euclidean geodesics.

**Theorem 13.1** (Theorem C)**.** *Let $s, s', t$ be vertices in a systolic complex $X$ such that $|st| = n, |s't| = n'$. Let $(r_k)_{k=0}^{n}, (r'_k)_{k=0}^{n'}$ be 1–skeleton geodesics such that $r_k \in \delta_k, r'_k \in \delta'_k$, where $(\delta_k), (\delta'_k)$ are Euclidean geodesics for $t, s$ and for $t, s'$ respectively. Then for all $0 \leq c \leq 1$ we have $|r_{\lfloor cn \rfloor} r'_{\lfloor cn' \rfloor}| \leq c|ss'| + C$, where $C$ is a universal constant.*

In the proof we need three easy preparatory lemmas.

**Lemma 13.2.** *Let $D$ be a 2–dimensional systolic complex (in particular $CAT(0)$ with the standard piecewise Euclidean metric). Let $x, y$ be vertices in $D$. Then there exists a 1–skeleton geodesic $\omega$ in $D$ joining $x, y$ such that if $D_0$ is the union with $\omega$ of a connected component of $D \setminus \omega$, then the $CAT(0)$ geodesic $xy$ in $D_0$ is 1–close to $\omega$.*

**Proof.** Let $L_i$ be the layers in $D$ between $x, y$ and let $L$ be the span in $D$ of the union of $L_i$. Observe that $L$ is convex in $CAT(0)$ sense in $D$. Hence the $CAT(0)$ geodesic $xy$ in $D$ is contained in $L$. Now similarly as in Definition 9.10 define vertices $\omega_i \in L_i$ to be the vertices nearest to $xy \cap L_i$ (possibly non-unique). Analogically as in Lemma 9.11 one proves that $\omega_i, \omega_{i+1}$ are neighbors, hence $(\omega_i)$ form a path $\omega$, which is a 1–skeleton geodesic. By the construction we have $|\omega_i, xy \cap L_i| \leq \frac{1}{2}$ (here $|\cdot,\cdot|$ denotes the distance along the straight line). For a fixed $D_0$ the $CAT(0)$ geodesic $xy$ in $D_0$ is contained in $L \cap D_0$, hence it is 1–close to $\omega$ by Lemma 12.12 applied to $L$. □

**Lemma 13.3.** *Let $\Delta$ be a generalized characteristic disc for $(i,j)$. Let $\Delta_{split} \subset \Delta$ be a generalized characteristic disc for $(i,j)$ with $w_k$ substituted with $\dot{w}_k$ for some $\dot{w}_k \in v_k w_k$. Let $\gamma, \dot{\gamma}$ be $CAT(0)$ geodesics with common endpoints in the layers $i,j$ in $\Delta, \Delta_{split}$, respectively. Then $\dot{\gamma} \cap v_k w_k$ is not farther from $v_k$ then $\gamma \cap v_k w_k$.*

**Proof.** Let $\Delta_0 \subset \Delta$ be the characteristic disc for $(i,j)$ with $w_k$ substituted with $\gamma \cap v_k w_k$. Then $\Delta_0 \cap \Delta_{split}$ is convex in $\Delta_{split}$ and we are done. □

**Lemma 13.4.** *Let $T$ be a $CAT(0)$ (i.e. simply connected) subspace of $\mathbb{E}^2$, whose boundary is an embedded loop which consists of three geodesic (in $T$) segments $\alpha, \beta, \gamma$, where $\alpha$ is contained in a straight line in $\mathbb{E}^2$. Denote $x = \beta \cap \gamma$. Let $\eta$ be a geodesic in $T$ contained in a straight line parallel to $\alpha$ with endpoints on $\beta, \gamma$. Let $c$ denote the ratio of the distances in $\mathbb{E}^2$ between $x$ and the line containing $\eta$ and between $x$ and the line containing $\alpha$. Then $\frac{|\eta|}{|\alpha|} \leq c$.*

**Proof.** Let $y_1, y_2 \in \mathbb{E}^2$ be points on the line containing $\eta$ colinear with $x$ and the endpoints of $\alpha$. By the Tales Theorem we have $\frac{|y_1 y_2|}{|\alpha|} = c$. On the other hand, since $\beta, \gamma$ are geodesics in $T$, we get that $\eta \subset y_1 y_2$. □

We are now ready for the endgame.

**Proof of Theorem 13.1 (Theorem C).** Let $m$ be maximal satisfying $r_m = r'_m$. First assume that $\lfloor cn \rfloor \leq m$ or $\lfloor cn' \rfloor \leq m$, say $\lfloor cn' \rfloor \leq m$. Then $|r_{\lfloor cn' \rfloor} r'_{\lfloor cn' \rfloor}| \leq 198$. Indeed, let $\Delta$ be the characteristic disc for $(r_i), (r'_i)$ between $t$ and $r_m = r'_m$, for the thick interval containing $\lfloor cn' \rfloor$ (if layer $\lfloor cn' \rfloor$ is thin then there is nothing to prove). Then by Proposition 12.1 applied to $(r_i)_{i=0}^n$ and $r'_0, \ldots r'_m, r_{m+1}, \ldots r_n$ we get that the $CAT(0)$ geodesic in $\Delta$ joining the barycenters of the two outermost edges is 99–close to the boundary component corresponding to $(r_i)$. Similarly we get that this $CAT(0)$ geodesic is 99–close to the second boundary component. Altogether we get that $|r_{\lfloor cn' \rfloor} r'_{\lfloor cn' \rfloor}| \leq 198$, as desired. This yields

$$
\begin{aligned}
|r_{\lfloor cn \rfloor} r'_{\lfloor cn' \rfloor}| \leq |r_{\lfloor cn \rfloor} r_{\lfloor cn' \rfloor}| + |r_{\lfloor cn' \rfloor} r'_{\lfloor cn' \rfloor}| \leq |\lfloor cn \rfloor - \lfloor cn' \rfloor| + 198 < \\
< c|n - n'| + 199 \leq c|ss'| + 199,
\end{aligned}
$$

as required. So from now on we assume that $\lfloor cn \rfloor > m$ and $\lfloor cn' \rfloor > m$.

Let $k$ be minimal such that $r_k$ lies on some 1–skeleton geodesic $ss'$. Now let $k'$ be minimal such that $r'_{k'}$ lies on some 1–skeleton geodesic $r_k s'$. Consider various 1–skeleton geodesics $\psi$ connecting $r_k$ with $r_{k'}$. The loops $r_m r_{m+1} \ldots r_k \psi r'_{k'} r'_{k'-1} \ldots r'_m$ are embedded by the choice of $m, k, k'$. Consider a surface $S \colon D \to X$ of minimal area spanned on such a loop (we allow $\psi$

to vary). By minimality of the area $D$ is systolic, hence $CAT(0)$ w.r.t. the standard piecewise Euclidean metric. Denote the preimages of $r_i, r'_i, \psi$ in $D$ by $x_i, x'_i, \alpha$ respectively. We attach to $D$ at $x_k, x'_{k'}, x_m = x'_m$ three simplicial paths $\beta, \beta', \zeta$ of lengths $n-k, n'-k', m$ respectively and denote obtained in this way simplical (and $CAT(0)$) complex by $D'$. Denote the vertices in $D' \setminus D$ by $x_n, \ldots, x_{k+1}$, by $x'_{n'}, \ldots, x'_{k'+1}$, and by $x_0 = x'_0, \ldots, x_{m-1} = x'_{m-1}$ in $\beta, \beta', \zeta$ respectively.

By minimality of the area of $D$, the path $\beta\alpha\beta'^{-1}$ is a $CAT(0)$ geodesic in $D'$. Let $D_1, D_2$ be simplicial spans in $D'$ of the unions of all 1–skeleton geodesics from $x_0$ to $x_n$ and from $x'_0$ to $x'_{n'}$ respectively. Observe that $D_1, D_2$ are convex (in $CAT(0)$ sense) in $D'$, hence the $CAT(0)$ geodesics in $D'$ from $x_0$ to $x_n$ and from $x'_0$ to $x'_{n'}$ agree with $CAT(0)$ geodesics joining those pairs of points in $D_1, D_2$, respectively. By Proposition 12.1, $(x_i)$ is 99–close (in $D_1$) to the $CAT(0)$ geodesic $x_0x_n$ and $(x'_i)$ is 99–close (in $D_2$) to the $CAT(0)$ geodesic $x'_0x'_{n'}$.

Our goal, which immediately implies Theorem 13.1 (Theorem C), is to get an estimate $|x_{\lfloor cn \rfloor}x'_{\lfloor cn' \rfloor}| \leq c|x_nx'_{n'}| + C$ with some universal constant $C$.

We claim that for any three consecutive vertices $v, w, u$ on $\alpha$ we have that $|x_0w| = |x_0v| + 1$ implies $|x_0u| = |x_0w| + 1$. We prove this claim by contradiction. If $|x_0u| = |x_0w| - 1$ then, by Lemma 2.8, $u, v$ are neighbors contradicting the fact that $vwu$ is a 1–skeleton geodesic. If $|x_0u| = |x_0w|$, then by Lemma 2.8 there exists a vertex $z \in D$ in the projection of the edge $wu$ onto $B_{|x_0v|}(x_0)$. Again by Lemma 2.8, we have that $|zv| \leq 1$. Thus the defect at $w$ is $\geq 1$, contradicting the minimality of the area of $D$. This justifies the claim.

The claim implies that $\alpha$ is a concatenation $\alpha_1\alpha_0\alpha_2$, where vertices in $\alpha_0$ are at constant distance from $x_0$ and $\alpha_1, \alpha_2$ are contained in 1–skeleton geodesic rays in $D'$ issuing from $x_0$. We apply Lemma 13.2 to obtain a special 1–skeleton geodesic $\omega$ in $D'$ connecting $x_0$ to $\alpha_1 \cap \alpha_0$. Let $\widetilde{D}_1$ be the union of $\omega$ and all of the components of $D' \setminus \omega$ containing some $x_i$ (i.e. on one "side" of $\omega$). Denote by $\widetilde{D}_1^c$ the union of $\omega$ with the other components of $D' \setminus \omega$. Denote by $\omega'$ a 1–skeleton geodesic connecting $x_0$ to $\alpha_0 \cap \alpha_2$ given by Lemma 13.2 applied do $\widetilde{D}_1^c$. Let $\widetilde{D}_2$ be the union of $\omega'$ with the components of $\widetilde{D}_1^c \setminus \omega'$ containing some $x'_i$. Denote the union of $\omega'$ with the other components of $\widetilde{D}_1^c \setminus \omega'$ by $\widetilde{D}_0$.

Note that, since $\widetilde{D}_1 \subset D_1, \widetilde{D}_2 \subset D_2$, by Lemma 13.3 we have that $(x_i)$ is 99–close to the $CAT(0)$ geodesic $x_0x_n$ in $\widetilde{D}_1$ and $(x'_i)$ is 99–close to the $CAT(0)$ geodesic $x'_0x'_{n'}$ in in $\widetilde{D}_2$. Moreover, by Lemma 13.2 and Lemma 13.3, the $CAT(0)$ geodesics in $\widetilde{D}_0, \widetilde{D}_1, \widetilde{D}_2$ joining the endpoints of $\omega, \omega'$ are 1–close (in particular 99–close) to $\omega, \omega'$, respectively. Moreover, vertices in $\alpha_0$ are at constant distance from $x_0$ in $\widetilde{D}_0$, and $\omega\alpha_1^{-1}, \omega'\alpha_2$ are 1–skeleton geodesics in $\widetilde{D}_1, \widetilde{D_2}$, respectively. Thus substituting $D' = \widetilde{D}_0, \widetilde{D}_1, \widetilde{D}_2$ we have reduced the proof of Theorem 13.1 (up to replacing $C$ with $3C$) to the following two special cases:

**(i)** vertices in $\alpha$ are at a constant distance from $x_0$ (hence from $x_m$) or
**(ii)** $n' = k'$ and $\alpha x'_{k'} \ldots x'_0$ is a 1–skeleton geodesic.

Observe that it is now possible that $x_i = x'_i$ for $i > m$. Let $m'$ be maximal such that $x_{m'} = x'_{m'}$. If $\lfloor cn \rfloor \leq m'$ or $\lfloor cn' \rfloor \leq m'$, say the latter, then, since the $CAT(0)$ geodesics $x_0x_n, x'_0x'_{n'}$ in $D'$ coincide on $x_0x_{m'}$, we get that $|x_{\lfloor cn' \rfloor}x'_{\lfloor cn' \rfloor}| \leq 99 + 99 = 198$, hence

$$|x_{\lfloor cn \rfloor}x'_{\lfloor cn' \rfloor}| \leq |x_{\lfloor cn \rfloor}x_{\lfloor cn' \rfloor}| + |x_{\lfloor cn' \rfloor}x'_{\lfloor cn' \rfloor}| \leq$$
$$\leq |\lfloor cn \rfloor - \lfloor cn' \rfloor| + 198 < c|n - n'| + 199 \leq c|x_nx'_{n'}| + 199,$$

as desired. So from now on we can assume that $\lfloor cn \rfloor > m', \lfloor cn' \rfloor > m'$, and we can replace the component of $D' \setminus x_{m'}$ containing $x_0$ with a simplicial path of length $m'$. Let $D$ be as before the maximal subcomplex of $D'$ which is a topological disc.

First suppose that we are in case (i). Observe that (up to increasing $C$ by 2) we can assume that $n = k$ and $n' = k'$. This is because once we proved our estimate for $n = k, n' = k'$ we can concatenate an estimate realizing path $x_{\lfloor ck \rfloor}x'_{\lfloor ck' \rfloor}$ with the paths $x_{\lfloor ck \rfloor} \ldots x_{\lfloor cn \rfloor}$ and $x'_{\lfloor ck' \rfloor} \ldots x'_{\lfloor cn' \rfloor}$, obtaining a path from $x_{\lfloor cn \rfloor}$ to $x'_{\lfloor cn' \rfloor}$ of length

$$(\lfloor cn \rfloor - \lfloor ck \rfloor) + |x_{\lfloor ck \rfloor}x'_{\lfloor ck' \rfloor}| + (\lfloor cn' \rfloor - \lfloor ck' \rfloor) <$$
$$< (c(n - k) + 1) + (c|x_kx_{k'}| + C) + (c(n' - k') + 1) =$$
$$= (c(|x_nx_k|) + 1) + (c|x_kx_{k'}| + C) + (c(|x_{n'}x_{k'}|) + 1) = c|x_nx_{n'}| + (C + 2),$$

as required.

We claim that $D$ is flat and the interior vertices of $\alpha$ have defect 0. Indeed, observe that the defects at the interior vertices of $\alpha$ and at the interior vertices of $D$ are $\leq 0$, whereas the defect at $x_{m'} = x'_{m'}$ is $\leq 2$. Hence, by Gauss–Bonnet Lemma 7.2, it is enough to prove that the sums of the defects at the vertices of each of the paths $x_{m'+1} \ldots x_k$ and $x'_{m'+1} \ldots x_{k'}$ are $\leq 2$. Suppose otherwise, w.l.o.g., that the sum of the defects at the vertices of $x_{m'+1} \ldots x_k$ is $\geq 3$. Denote the vertex following $x_k$ on $\alpha$ by $y$. Then $|x_{m'}y| \leq |x_{m'+1}x_k|$, hence $|x_0y| < |x_0x_k|$, which contradicts the assumptions of case (ii). Thus we have proved the claim. In particular, $\alpha$ is contained in a straight line in $D \subset \mathbb{E}^2_\Delta$ and $k = k'$.

Define $\eta$ to be the path in $D$ starting at $x_{\lfloor ck \rfloor}$ reaching $x'_{\lfloor ck \rfloor}$ contained (in $D \subset \mathbb{E}^2_\Delta$) in a straight line parallel to $\alpha$. Let $\xi_1, \xi_2$ be $CAT(0)$ geodesics in $D$ joining $x_k$ with $x_{m'}$ and $x'_k$ with $x'_{m'} = x_{m'}$, respectively. Let $z_i = \eta \cap \xi_i$, for $i = 1, 2$. We have $|x_{\lfloor ck \rfloor} z_1| \leq 99$ and $|z_2 x'_{\lfloor ck \rfloor}| \leq 99$ (again exceptionally $|\cdot, \cdot|$ denotes the distance along the straight line). Let $m''$ be maximal such that $\xi_1 \cap x_{m''} x'_{m''} = \xi_2 \cap x_{m''} x'_{m''}$. Then for all $i \leq m''$ we have $\xi_1 \cap x_i x'_i = \xi_2 \cap x_i x'_i$. In particular, if $\lfloor ck \rfloor \leq m''$, then $z_1 = z_2$ and $|\eta| \leq 198$, as desired. If $\lfloor ck \rfloor > m''$, then we apply Lemma 13.4 with $T \subset D$ the geodesic triangle with vertices $x_k, x'_k, \xi_1 \cap x_{m''} x'_{m''} = \xi_2 \cap x_{m''} x'_{m''}$. We get that $|\eta| \leq c|x_k x'_k| + 198$, as desired.

Now suppose that we are in case (ii). Like in case (i) (up to increasing $C$ by 1) we can assume that $n = k$. Since the boundary of $D$ is a union of two geodesics, by Gauss–Bonnet Lemma 7.2, $D$ is flat. Consider an embedding $D \subset \mathbb{E}^2_\Delta$ such that the layers (denoted by $L_k$) between $x_{m'} = x'_{m'}$ and $x_k$ in $\mathbb{E}^2_\Delta$ are horizontal and $x_i$ are to the left from $x'_i$, for $i \leq k'$. By minimality of area, $\alpha$ is contained in a straight line in $D \subset \mathbb{E}^2_\Delta$. Like in case (i), let $\xi_1, \xi_2$ be $CAT(0)$ geodesics in $D$ joining $x_k$ with $x_{m'}$ and $x'_{k'}$ with $x'_{m'} = x_{m'}$, respectively. Similarly like in the previous case, let $m''$ be maximal such that $\xi_1 \cap L_{m''} = \xi_2 \cap L_{m''}$. Denote $u = \xi_1 \cap L_{m''} = \xi_2 \cap L_{m''}$. By the same argument as after the choice of $m'$, we can assume that $\lfloor ck' \rfloor > m''$. Let $z_1 = \xi_1 \cap L_{\lfloor ck \rfloor}$, $z_2 = \xi_2 \cap L_{\lfloor ck' \rfloor}$. Let $y_1 \in L_{\lfloor ck \rfloor} \cap D$ be the vertex with minimal possible $y_1^x$ but $\geq z_1^x$. Similarly, let $y_2 \in L_{\lfloor ck' \rfloor} \cap D$ be the vertex with maximal possible $y_2^x$ but $\leq z_2^x$. We claim that $|y_1 y_2| = \lfloor ck \rfloor - \lfloor ck' \rfloor$.

Before we justify the claim, observe that it already implies the theorem. Indeed, the claim gives

$$\begin{aligned}
|x_{\lfloor ck \rfloor} x'_{\lfloor ck' \rfloor}| &\leq |x_{\lfloor ck \rfloor} y_1| + |y_1 y_2| + |y_2 x'_{\lfloor ck' \rfloor}| \leq \\
&\leq 99 + (\lfloor ck \rfloor - \lfloor ck' \rfloor) + 99 < \\
&< 99 + (c(k' - k) + 1) + 99 = c|x_k x'_{k'}| + 199,
\end{aligned}$$

as desired.

Finally, let us justify the claim. We need to show that $y_2^x - y_1^x \leq \frac{\lfloor ck \rfloor - \lfloor ck' \rfloor}{2}$. By the choice of $m''$ we have that $z_1, z_2$ lie in the Euclidean triangle in $\mathbb{E}^2_\Delta$ with vertices $x_k, x'_{k'}, u$. Denote by $u_1$ (resp. $u_2$) the vertex on the edge $u x_k$ (resp. $u x'_{k'}$) of this triangle in $L_{\lfloor ck \rfloor}$ (resp. $L_{\lfloor ck' \rfloor}$). Assume w.l.o.g. that $\frac{\lfloor ck' \rfloor}{k'} \geq \frac{\lfloor ck \rfloor}{k}$. Denote then by $u_*$ the vertex on the edge $u x_k$ dividing this edge in same proportion as the proportion in which $u_2$ divides $u x'_{k'}$. By the Tales Theorem, and since $u_1 u_* \subset u x_k$ forms with the vertical direction angle

$\leq 30°$, we have that

$$u_2^x - u_1^x \leq (u_2^x - u_*^x) + (u_*^x - u_1^x) < c\big(x_k^x - (x'_{k'})^x\big) + \frac{1}{2} =$$
$$= \frac{ck - ck'}{2} + \frac{1}{2} < \frac{\lfloor ck \rfloor - \lfloor ck' \rfloor}{2} + 1,$$

hence

$$y_2^x - y_1^x \leq z_2^x - z_1^x \leq u_2^x - u_1^x < \frac{\lfloor ck \rfloor - \lfloor ck' \rfloor}{2} + 1.$$

Thus, since $y_2^x - y_1^x$ and $\frac{\lfloor ck \rfloor - \lfloor ck' \rfloor}{2}$ differ by an integer (because $y_1, y_2$ are vertices in $\mathbb{E}^2_\Delta$), we have $y_2^x - y_1^x \leq \frac{\lfloor ck \rfloor - \lfloor ck' \rfloor}{2}$, as desired. This ends the proof of the claim and of the whole theorem. □

If we followed the constants carefully, we would get that Theorem 13.1 (Theorem C) is satisfied with any $C \geq 208$.

# 14 Final remarks

In this section we state some additional results on the compactification $\overline{X}$, for which we do not provide proofs.

$EZ$–structures explored by Farrell–Lafont [13] in relation to the Novikov conjecture concern only the torsion–free group case. To get similar results (Novikov conjecture) for a group $G$ with torsion one needs to construct an appropriate compactification (which we will also call an $EZ$–structure) of a *classifying space for proper $G$–actions*, denoted $\underline{E}G$. $\underline{E}G$ is a contractible space with a proper $G$ action such that, for every finite subgroup $F$ of $G$, the set $\underline{E}G^F \subset \underline{E}G$ of points fixed by $F$ (the *fixed point set* of $F$) is contractible (in particular non–empty).

For a systolic group $G$ acting geometrically on a systolic complex $X$ it is possible, that $X$ is an $\underline{E}G$. This is not known yet however, due to the fact that it is not known whether the fixed point theorem holds for finite groups acting on systolic complexes. The best result in this direction is the theorem of Przytycki [19] saying that, for every finite group $F$ acting on a systolic complex $X$, there is a non–empty $F$–invariant subcomplex of $X$ of diameter at most 5. Using this, Przytycki proved [20] that the Rips complex $R_5(X)$ of $X$ is an $\underline{E}G$. Thus we can get the desired $EZ$–structure by compactifying $R_5(X)$, using the following result analogous to [2, Lemma 1.3] (and whose proof follows the lines of the proof of the latter).

**Lemma 14.1.** *Let $(\overline{X}, \partial X)$ be an EZ–structure on G and let G act geometrically on a contractible space Y. Then there is a natural EZ–structure $(Y \cup \partial X)$ on G.*

Thus in our case we get a compactification $\overline{R_5(X)} = R_5(X) \cup \partial X$ of $R_5(X)$ by adjoining our boundary $\partial X$ of $X$ to the Rips complex $R_5(X)$. We claim that the following holds.

**Claim 14.2.** *Let a group G act geometrically by simplicial automorphisms on a systolic complex X. Let $\overline{R_5(X)} = R_5(X) \cup \partial X$ be the compactification of $R_5(X)$ obtained by applying Lemma 14.1 to $(\overline{X}, \partial X)$. Then the following hold:*

1. *$\overline{R_5(X)}$ is a Euclidean retract (ER),*
2. *$\partial X$ is a Z–set in $\overline{R_5(X)}$,*
3. *for every compact set $K \subset R_5(X)$, $(gK)_{g \in G}$ is a null sequence,*
4. *the action of G on $R_5(X)$ extends to an action, by homeomorphisms, of G on $\overline{R_5(X)}$,*
5. *for every finite subgroup F of G, the fixed point set $\overline{R_5(X)}^F$ is contractible,*
6. *for every finite subgroup F of G, the fixed point set $R_5(X)^F$ is dense in $\overline{R_5(X)}^F$.*

Assertions 1–4 follow from Lemma 14.1. Assertion 6 is also easy to prove. The only difficulties in proving Claim 14.2 concern assertion 5. To obtain it one has to introduce good geodesics in the Rips complex and reprove Lemma 6.2 with $\overline{R_5(X)}^F$ in place of $\overline{X}$.

Combining Claim 14.2 above and Theorem 4.1 of Rosenthal [21], we immediately get the following.

**Claim 14.3.** *The Novikov conjecture holds for systolic groups.*

Note that if the fixed point theorem holds for finite groups acting on $X$, then, by [20], we have that $X$ is $\underline{E}G$. Then in Claim 14.2 we can substitute $\overline{R_5(X)}$ with $\overline{X}$ and it is easier to prove assertion 5 in this case. Then we can apply Theorem 4.1 from [21] directly to $\overline{X}$ to obtain Claim 14.3.

Now we turn to the question of determining our boundary in some specific cases. We have already mentioned the case of hyperbolic systolic groups in Remark 4.5. Now we consider the $CAT(0)$ case. After making it through the second part of the article, the reader should not be surprised by the following.

**Claim 14.4.** *If $X$ is a two–dimensional simplicial complex, which is $CAT(0)$ (which is equivalent with systolic in dimension two), then its compactification by the $CAT(0)$ visual boundary is homeomorphic in a natural way with our $\overline{X}$.*

For example, this implies that our boundary of a systolic Euclidean plane is a circle. The argument for Claim 14.4 is that our compactification is constructed using Euclidean geodesics in systolic complexes, which in this case are coarsely $CAT(0)$ geodesics.

The next claim concerns the following construction, which has not yet appeared in the literature. Namely Elsner and Przytycki had developed a way to turn equivariantly any $\mathcal{VH}$–complex which is $CAT(0)$ into a systolic complex (that is how they observed that the abelian product of two free groups is systolic). Although the resulting complex is usually not 2–dimensional, the only higher dimensional simplices that appear are used to deal with branching at the vertical edges. This is why we believe that the $CAT(0)$ visual boundary of the original $\mathcal{VH}$–complex is homeomorphic in a natural way with our boundary of the resulting systolic complex.

In particular, this would imply that there is a systolic group acting geometrically on two systolic complexes whose (our) boundaries are not homeomorphic. Namely, in the family of torus complexes defined by Croke–Kleiner [7] the complexes with $\alpha = \frac{\pi}{2}$ and $\alpha = \frac{\pi}{3}$ have universal covers with non–homeomorphic $CAT(0)$ visual boundaries. At the same time, there is a torus complex with $\alpha = \frac{\pi}{3}$, whose universal cover is 2–dimensional systolic while there also is a torus complex with $\alpha = \frac{\pi}{2}$, whose universal cover is a $\mathcal{VH}$–complex, which is $CAT(0)$.

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