Filling invariants of systolic complexes and groups

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Abstract

Systolic complexes are simplicial analogues of nonpositively curved spaces. Their theory seems to be largely parallel to that of CAT(0) cubical complexes.

We study the filling radius of spherical cycles in systolic complexes, and obtain several corollaries. We show that a systolic group can not contain the fundamental group of a nonpositively curved riemannian manifold of dimension strictly greater than 2, although there exist word hyperbolic systolic groups of arbitrary cohomological dimension.

We show that if a systolic group splits as a direct product, then both factors are virtually free. We also show that systolic groups satisfy linear isoperimetric inequality in dimension 2.

AMS Classification numbers Primary: 20F69, 20F67

Secondary: 20F65

Keywords: systolic complex, systolic group, filling radius, word-hyperbolic group, asymptotic invariant

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1 Introduction

Systolic complexes were introduced in Januszkiewicz–Świątkowski [12] and, independently, in Haglund [9]. They are simply connected simplicial complexes that satisfy certain condition that we call simplicial nonpositive curvature (abbreviated SNPC). The condition is local and purely combinatorial. It neither implies nor is implied by nonpositive curvature for geodesic metrics on complexes, but it has many similar consequences. It is a simplicial analogue of the combinatorial condition for cubical complexes equivalent to nonpositive curvature for the standard piecewise euclidean cubical metric, namely the flag property for links at vertices.

SNPC complexes of groups are developable. This allowed us to construct in [12] numerous examples of interesting systolic complexes of arbitrary dimension (pseudomanifolds, normal chamber complexes of arbitrary thickness) with large automorphism groups. These groups led to the notion of a systolic group, defined in [12] as groups that act by simplicial automorphisms, properly discontinuously and cocompactly on a systolic complex. It was shown that systolic groups are biautomatic. Their existence was established in arbitrary cohomological dimension. A minor modification of the construction yields also groups that are in addition word-hyperbolic.

Ideas related to systolic complexes allowed us to answer various open questions. For example, we isolated a simple combinatorial condition for simplicial complexes that implies word-hyperbolicity of their fundamental groups, and which works in dimensions greater than 2. Another simple combinatorial condition turned out to be sufficient for a simplicial complex to be $CAT(0)$ with respect to the standard piecewise euclidean metric. Constructions using systolic complexes yield finite nonpositively curved (in metric sense) branched coverings of arbitrary finite simplicial piecewise euclidean pseudomanifolds. They also provide finite nonpositively curved developments of euclidean simplicial billiard tables in arbitrary dimension. The reader is referred to Januszkiewicz–Świątkowski [12] for details concerning these results.

Further properties of systolic groups were found by D. Wise [18]. For example, he showed that every finitely presented subgroup of a systolic group is systolic. He also found interesting connections between the systolic ideas and small cancellation theory.

In the present paper, which is a continuation of [12], we study filling properties of spherical cycles in systolic complexes and groups. One of our motivations was to show that in the systolic world one meets objects very different from classical ones. The following result is a manifestation of this phenomenon.

Theorem A (see Corollary 6.4 in the text) *Let G be a systolic group. Then G contains no subgroup isomorphic to the fundamental group of a closed non-positively curved riemannian manifold of dimension strictly greater than 2.*

Theorem A together with the construction in [12] of word-hyperbolic systolic groups of arbitrary cohomological dimension yields the following.

Corollary B *For each natural number $n > 2$ there is a word-hyperbolic group of cohomological dimension n containing no subgroup isomorphic to the fundamental group of a closed nonpositively curved riemannian manifold of dimension greater than 2.*

It is interesting to contrast Corollary B with conjectural existence of a surface subgroup in every word-hyperbolic group of dimension greater than 1.

The above results are related to an informal conjecture of M. Gromov saying that every construction of hyperbolic groups of large (rational) cohomological dimension uses arithmetic groups as building blocks. A more precise form of this conjecture, stated as Question Q 1.3 in Bestvina [1], asks whether for every $K > 0$ there is $N > 0$ such that every word-hyperbolic group G of rational cohomological dimension $\geq N$ contains an arithmetic lattice of dimension $\geq K$. Examples of word-hyperbolic systolic groups of large cohomological dimension, constructed in [12], show that this is not true. Cocompact lattices (of large cohomological dimension) are excluded by Theorem A, while cofinite volume ones are excluded by biautomaticity.

Remark We would like to mention here that hyperbolic Coxeter groups of large dimension, constructed in [11], as well as the fundamental groups of their nerves, are also systolic, and hence satisfy the conclusion of Corollary B. This requires some additional arguments, which we omit.

Remark Damian Osajda [14] has obtained a partial improvement of Theorem A, see comment 6.7.2.2

The second main result of the paper deals with systolic groups that are products, showing that there are very few of them.

Theorem C (see Corollaries 8.5 and 8.7 in the text) *If the product of two infinite groups is systolic then both factor groups are virtually free. The product of more than two infinite groups is never systolic.*

Theorems A and C are proved using the concept of the *filling radius for spherical cycles*. Roughly speaking, a k-spherical cycle in a space is a map from the sphere S^k to (an appropriate thickening of) the space, and its *filling* is a singular chain in the space bounded by this cycle. The *filling radius* of a spherical cycle C is the infimum over all fillings D of C of the maximal distance of a point in D from C. Typically, the filling radius of C grows with the size (e.g. diameter) of C. Strikingly, in systolic complexes the filling radius of k-spherical cycles, for $k \geq 2$, is universally bounded from above. More precisely, we have the following.

Lemma (see Lemma 4.4 in the text) *Let $f : S \to X$ be a simplicial map from a triangulation S of the sphere S^k, for some $k \geq 2$, to a systolic complex X. Then f has a simplicial extension $F : B \to X$, for some triangulation B of the ball B^{k+1} with $\partial B = S$, such that the image $F(B)$ is contained in the full subcomplex of X spanned by the image $f(S)$.*

In Section 5 we introduce an asymptotic property of metric spaces inspired by the above lemma. We call this property S^kFRC, which is an abbreviation of the phrase "k-spherical cycles have filling radius constant (i.e. uniformly bounded)". In fact, this gives a sequence of properties, one for each natural number k. Properties S^kFRC are shown to be preserved by quasi-isometries and hence applicable to finitely generated groups equipped with word metrics. Theorem A above is a consequence of the following result concerning S^kFRC.

Theorem D

1. (see Corollary 3.4 in combination with Lemma 5.3) *If for some natural number k a group G is S^kFRC, then any subgroup $H < G$ is S^kFRC.*

2. (see Theorem 4.1 in combination with Lemma 5.3) *Systolic groups are S^kFRC for any $k \geq 2$.*

3. (see Proposition 6.1) *Let G be the fundamental group of a closed nonpositively curved riemannian manifold of dimension n. Then G is not S^kFRC for $1 \leq k \leq n-1$.*

We think it is worth emphasizing that among numerous filling invariants it is the filling radius (and not the more commonly used filling volume) that works particularly efficiently in the context of systolic complexes and groups. The next two facts concerning this invariant imply the second main result of the paper, Theorem C above.

Theorem E (see Proposition 7.1 in the text) *The product of more than two infinite groups is never S^2FRC. If the product of two infinite groups is S^2FRC, then both factor groups are S^1FRC.*

Theorem F (see Propositions 8.1 and 8.2 in the text) *If a finitely presented group G is S^1FRC, then it is word-hyperbolic and its Gromov boundary has topological dimension 0. Consequently, G is virtually free.*

As a byproduct of our arguments we obtain the following corollary about groups with asymptotic dimension 1. For a background concerning asymptotic dimension the reader is referred to Roe [16]. The Corollary follows from Theorem F and Proposition 5.4.

Corollary G *A finitely presented group with asymptotic dimension 1 is virtually free.*

One more idea contained in the paper and related to filling radius is the concept of *asymptotic hereditary asphericity* (AHA) introduced in Section 3. It implies S^kFRC for all $k \geq 2$ and it is satisfied by systolic complexes and groups. As the name suggests, it is inspired by the property of hereditary asphericity for topological spaces (see Davermann [4] and Davermann–Dranishnikov [5]). AHA (and in fact its minor strengthening, see comment 4.6.1) is the strongest presently known geometric (i.e. quasi-isometry invariant) property of systolic complexes and groups. It is inherited by uniformly embedded subspaces and in particular by arbitrary finitely generated subgroups. In this paper we use AHA only to shorten exposition, but we suspect it may play a role in future developments.

The last section of the paper deals with filling volume in systolic complexes. The exposition is essentially independent of the previous sections and the proofs use different methods. The main result, Theorem H below, was suggested to us by D. Wise. The reader is referred to Section 9 for the precise statement of this result.

Theorem H (see Theorem 9.2 in the text) *The filling volume for 2-spherical cycles in systolic complexes is linearly bounded by the volume of the cycles.*

We finish each of the sections 3–9 with a subsection that contains comments which complement the content of a section, and/or some open questions.

Acknowledgments. The first author was partially supported by the NSF grant DMS-0405825. The second author was partially supported by the KBN grant 2 P03A 017 25. We thank The Ohio State University Research Foundation for partial support of visits of the second author to OSU.

2 Systolic complexes and groups — background

In this section we recall definitions, basic properties and some results concerning systolic complexes and groups. Our main reference is Januszkiewicz and Świątkowski [12].

A simplicial complex X is *flag* if every finite subset of its vertices pairwise connected by edges spans a simplex of X. A *cycle* in a simplicial complex is a subcomplex homeomorphic to the circle S^1. The *length* $|\gamma|$ of a cycle γ is the number of its edges. We say that a cycle in X has a *diagonal* if some two nonconsecutive vertices in this cycle are connected by an edge in X.

Given a natural number $k \geq 5$, a simplicial complex X is k*-large* if it is flag and every cycle in X of length $4 \leq |\gamma| < k$ has a diagonal.

A simplicial complex X is *locally* k*-large* if its links at all (nonempty) simplices are k-large. X is k*-systolic* if it is locally k-large, connected and simply connected. Since the case of $k = 6$ is particularly important, we abbreviate the term "6-systolic" to an easier to pronounce term *systolic*.

We view local k-largeness (i.e., k-largeness of links) as a kind of curvature bound from above. We sometimes call local 6-largeness *simplicial nonpositive curvature* (abbreviated SNPC), as it yields similar consequences as metric nonpositive curvature. A systolic complex is then the simplicial analogue of what is called a $CAT(0)$ or Hadamard space.

We now list several straightforward properties of the above notions.

2.1 Fact

(1) If $k > m$ and X is k-large then X is also m-large.

(2) Any full subcomplex of a k-large simplicial complex is k-large.

(3) Links of a k-large complex are k-large, i.e. any k-large complex is locally k-large.

(4) If γ is a cycle of length $|\gamma| = m < k$ in a k-large simplicial complex X, then γ bounds in X a subcomplex Δ homeomorphic to the 2-disc consisting of $m-2$ 2-simplices (and thus containing no other vertices than the vertices of γ).

Given a vertex v in a simplicial complex X, let *balls in X centered at v* be the subcomplexes of X defined inductively as follows. Let $B_0(v, X) = v$ and put $B_{n+1}(v, X)$ to be the union of all those simplices of X that intersect $B_n(v, X)$. As a direct consequence of [12, Corollaries 7.5 and 7.4(2)] we have the following.

2.2 Lemma *If X is a systolic simplicial complex and v is any vertex of X then for every integer $n \geq 0$ the ball $B_n(v, X)$ is a deformation retract of the ball $B_{n+1}(v, X)$.*

This lemma has several important consequences gathered in the following.

2.3 Corollary

(1) (see [12, Corollary 7.5 and Lemma 7.2(1)]) *Balls in systolic complexes are contractible.*

(2) (see [12, Theorem 4.1(1)]) *Systolic simplicial complexes are contractible.*

(3) Locally 6-large simplicial complexes are aspherical (i.e. their universal covers are contractible). In particular, 6-large simplicial complexes are aspherical.

(4) Every full subcomplex in a 6-large simplicial complex is aspherical.

Corollary 2.3(2) above is an analogue, for simplicial curvature, of Cartan-Hadamard Theorem.

Let us mention that there is no obvious relationship between local 6-largeness (SNPC) and nonpositive curvature in metric sense. However, k-systolicity for k sufficiently large implies metric nonpositive curvature (see [12, Section 14] for details). Also, 7-systolicity of a simplicial complex implies its hyperbolicity in the sense of Gromov. More precisely, we have the following.

2.4 Theorem (see [12, Theorem 2.1]) *Let X be a 7-systolic simplicial complex. Then the 1-skeleton of X with its standard geodesic metric is δ-hyperbolic with $\delta = 2\frac{1}{2}$.*

A group is *k-systolic* if it acts by simplicial automorphisms, properly discontinuously and cocompactly, on a k-systolic simplicial complex. A group is *systolic*, if it is k-systolic with $k = 6$. In [12, Section 13] we have established the following.

2.5 Theorem *Every systolic group is biautomatic.*

The reader is referred to Epstein et al. [7] for the definition and properties of biautomatic groups. Biautomaticity has several algorithmic and geometric consequences. For example, every biautomatic group satisfies quadratic isoperimetric inequality and hence has solvable word problem. Moreover, every solvable subgroup of a biautomatic group is virtually abelian and every abelian subgroup is undistorted.

As a consequence of Theorem 2.4 above we also have the following property of groups.

2.6 Corollary *Every 7-systolic group is word-hyperbolic.*

Systolic complexes and groups exist in any dimension. The constructions from [12, Sections 18–20], as well as these of [11] imply, among others, the following.

2.7 Theorem

(1) (see [12, Corollary 19.2]) *For each natural number n and each $k \geq 6$ there exists an n-dimensional orientable compact simplicial pseudomanifold that is k-large.*

(2) (see [12, Corollary 19.3(1)]) *For each natural number n and every $k \geq 6$ there exists a developable simplex of groups whose fundamental group is k-systolic and has virtual cohomological dimension n.*

The fundamental group of a pseudomanifold as in Theorem 2.7(1) is easily seen to be torsion-free and to have cohomological dimension n. In both parts (1) and (2), if $k \geq 7$ then the corresponding fundamental group is in addition word-hyperbolic.

To formulate results showing even more flexibility in our constructions we need several definitions. An n-dimensional simplicial complex X is a *chamber complex* if it is the union of its n-simplices (which are the *chambers* of X). Clearly, links of a chamber complex are chamber complexes (of lower dimension). A chamber complex is *gallery connected* if any two of its chambers can be connected by a finite sequence of chambers such that any two consecutive chambers in this sequence share a face of codimension 1. A chamber complex is *normal* if it is gallery connected and all of its links are gallery connected.

Thickness of a chamber complex X at a face σ of codimension 1 is the number of chambers of X containing σ. For example, X is a pseudomanifold if its thickness is uniformly 2. Chamber complex X is *thick* if its thickness at every face of codimension 1 is greater than 2. *Thickness* of X is its maximal thickness at some codimension 1 face. The next result follows directly from [12, Proposition 19.1].

2.8 Theorem *For each natural number n and every $k \geq 6$ there is a compact n-dimensional thick chamber complex X, with arbitrarily large thickness (uniform or variable as a function on the set of codimension 1 faces) which is k-large.*

A simplicial map $f : X \to Y$ between pseudomanifolds is a *branched covering* if it is nondegenerate and its restriction to the complements of codimension 2 skeleta in X and Y is a covering.

2.9 Theorem (see [12, Proposition 20.3]) *Given any $k \geq 6$, every finite family of normal compact pseudomanifolds of the same dimension has a common compact branched covering which is k-large.*

3 Asymptotic hereditary asphericity

In this section we introduce the concept of asymptotic hereditary asphericity. We also show that it is invariant under uniform embeddings, and hence under quasi-isometries. In particular, when applied to finitely generated groups with word metrics, it passes to arbitrary finitely generated subgroups.

Recall that, given a metric space X and a real number $r > 0$, the *Rips complex* $P_r(X)$ is the simplicial complex with X as the vertex set, in which a finite subset of X spans a simplex if and only if all distances between the points in this subset are not greater than r. Clearly, if $0 < r < R$ then $P_r(X)$ is a subcomplex of $P_R(X)$.

We view any subset A of a metric space X as a metric space equipped with the restricted metric.

3.1 Definition

1. Given real numbers $0 < r \leq R$, a metric space X is (r,R)-*aspherical* if for every triangulation S of the sphere S^k, with $k \geq 2$, any simplicial map $f : S \to P_r(X)$ has a simplicial extension $F : B \to P_R(X)$ for some triangulation B of the ball B^{k+1} such that $\partial B = S$.

2. A metric space X is *asymptotically hereditarily aspherical* (shortly *AHA*) if for every $r > 0$ there is $R \geq r$ such that each subset $A \subset X$ is (r,R)-aspherical.

Recall that a (not necessarily continuous) map $h : (X_1, d_1) \to (X_2, d_2)$ between metric spaces is a *uniform embedding* if there are real functions g_1, g_2 with $g_i(t) \to +\infty$ as $t \to +\infty$, such that

$$g_1(d_1(x,y)) \leq d_2(h(x), h(y)) \leq g_2(d_1(x,y))$$

for all $x, y \in X_1$.

3.2 Proposition *Suppose that a metric space X_1 uniformly embeds in a metric space X_2 which is AHA. Then X_1 is AHA.*

Before giving a proof, we derive some consequences from Proposition 3.2. First, note that since a quasi-isometry is a uniform embedding, we have the following.

3.3 Corollary *Asymptotic hereditary asphericity is a quasi-isometry invariant.*

By Corollary 3.3, it makes sense to speak of AHA property for finitely generated groups, by requiring it to hold for the word metric associated to any finite generating set. Since a finitely generated subgroup of a finitely generated group uniformly embeds in this group (via inclusion), we have the following.

3.4 Corollary *If a finitely generated group Γ is AHA then every finitely generated subgroup of Γ is AHA.*

Proof of Proposition 3.2 Without loss of generality, we can assume that the functions g_1, g_2 in the estimates for the uniform embedding $h : X_1 \to X_2$ are monotone and that $g_1 \leq g_2$ (otherwise we replace them with $\hat{g}_1(t) := \inf_{s \geq t} g_1(s)$ and $\hat{g}_2(t) := \sup_{s \leq t} g_2(s)$).

Fix any subset $A \subset X_1$, any $r > 0$, and any simplicial map $f : S \to P_r(A)$ from a triangulated sphere S^k, for some $k \geq 2$. The map h induces the simplicial map $h_r : P_r(X_1) \to P_{g_2(r)}(X_2)$, and we consider the subset $h(A) \subset X_2$ and the composed map $h_r \circ f : S \to P_{g_2(r)}[h(A)]$. Since X_2 is AHA, there is $R' \geq g_2(r)$ (depending only on h and r, and not on A and f) and a simplicial extension $F' : B \to P_{R'}(h(A))$ of $h_r \circ f$, for some triangulation B of the ball B^{k+1} such that $\partial B = S$. Put $R := \sup\{t : g_1(t) \leq R'\}$ and note that, since $g_1(r) \leq g_2(r) \leq R'$, we have $R \geq r$. Moreover, if h is injective, the inverse map $h^{-1} : h(A) \to A$ induces the simplicial map $(h^{-1})_{R'} : P_{R'}[h(A)] \to P_R(A)$. The composed map $F = (h^{-1})_{R'} \circ F' : B \to P_R(A)$ is an extension of f showing that A is (r, R)-aspherical. Since the constant R above does not depend on A, the proposition follows.

In the case when h is not injective, we need to modify slightly the last part of the argument. Namely, we define a map F on the vertex set of B by requiring that $F(v) \in h^{-1}(F'(v))$ and $F(v) = f(v)$ for $v \in \partial B = S$. These conditions are compatible, since $h \circ f(v) = F'(v)$ for $v \in \partial B = S$. Moreover, since $d_2(F'(v), F'(w)) \leq R'$ for any two adjacent vertices in B, by definitions of R and F we have $d_1(F(v), F(w)) \leq R$ for any two such vertices. Therefore, F induces the simplicial map $B \to P_R(A)$ which is an extension of f, and the rest of the argument goes as at the end of the previous paragraph. □

3.5 Comments and questions

1. The property of asymptotic hereditary asphericity introduced in this section is inspired by the property of hereditary asphericity for topological (metric) spaces, see Davermann [4]. The latter was used to construct examples of spaces for which the cell-like maps do not raise dimension. Hereditarily aspherical spaces can have arbitrarily large dimension, see Davermann–Dranishnikov [5].

2. Recall that according to the (unresolved) Whitehead's conjecture every subcomplex of an aspherical 2-complex is aspherical. This provides the context for the following observations.

2.1. Let X be a complex of dimension ≤ 2 satisfying Whitehead's conjecture. Using methods similar to those in Section 4 it is not hard to show that $G = \pi_1 X$ is AHA.

2.2. $CAT(0)$ complexes of dimension ≤ 2 satisfy Whitehead's conjecture and hence are AHA. As a consequence, groups acting properly discontinuously and cocompactly on such complexes are AHA. Thus, product of two (virtually) free groups is AHA.

2.3. It follows from 2.2 above that virtually free groups are AHA. It is also not hard to see that, more generally, groups Γ with asdim$\Gamma \leq 1$ are AHA.

2.4. Whitehead's conjecture is satisfied by various classes of small cancellation complexes. For example, every subcomplex of a $C'(\frac{1}{6})$ complex is $C'(\frac{1}{6})$, and hence aspherical. Consequently, $C'(\frac{1}{6})$ small cancellation groups are AHA. Note that this class contains many groups that are not finitely presented.

3. Are there any 2-dimensional groups that are not AHA? This question may be viewed as group theoretic asymptotic variant of Whitehead's conjecture. Here dimension can have various meanings (virtual cohomological dimension, asymptotic dimension, etc.). Special cases of this question are the following.

3.1. Are word-hyperbolic groups with 1-dimensional boundary AHA?

3.2. Random groups obtained from the density model of Gromov [10] satisfy certain strong isoperimetric property, which allows to show that Whitehead's conjecture holds for their presentation complexes. Thus these groups are AHA. Are other natural classes of random groups (e.g., random quotients of hyperbolic groups) AHA? Are generic groups of Champetier [3] AHA?

4. Is a generic (in any reasonable sense) group of dimension ≥ 3 AHA?

5. There are AHA groups with both virtual cohomological and asymptotic dimensions arbitrarily large. These are for example hyperbolic systolic groups of arbitrarily large *vcd* constructed in Januszkiewicz–Świątkowski [12] (compare Theorem 2.7(2)). Their asymptotic dimension is large since for word-hyperbolic groups we have asdim$\Gamma \geq \dim \partial\Gamma + 1 = vcd\Gamma$ (see Bestvina–Mess [2] and Świątkowski [17]).

6. Damian Osajda [15] has proved that Gromov boundary of a 7-systolic group is strongly hereditary aspherical. We have reasons to believe that the same is true for all hyperbolic AHA groups. Is the converse true, i.e. is a word-hyperbolic group with hereditarily aspherical Gromov boundary necessarily AHA?

7. For what classes of subgroups H is the free product with amalgamation along H of AHA groups again AHA? Is this true for H finite, virtually cyclic, of asymptotic dimension 1, AHA? What about HNN-extensions?

4 Systolic complexes and groups are AHA

This section is entirely devoted to the proof of the following theorem and to the statement of its corollaries.

4.1 Theorem *Every systolic simplicial complex is asymptotically hereditarily aspherical. More precisely, for each $r > 0$, every subset A of the vertex set $X^{(0)}$ of a systolic simplicial complex X, with the induced from X polygonal distance, is $(r, 8r+17)$-aspherical.*

Before getting to the proof, we list some corollaries. Recall that a group is systolic if it acts simplicially properly discontinuously and cocompactly on a systolic simplicial complex. Since such a group, equipped with a word metric, is clearly quasi-isometric to the corresponding systolic complex, in view of Corollary 3.3 we obtain the following.

4.2 Corollary *Every systolic group is asymptotically hereditarily aspherical.*

Due to Corollary 3.4 we have also the following stronger result.

4.3 Corollary *Every finitely generated subgroup of a systolic group is AHA.*

We now turn to the proof of Theorem 4.1. The first step is the following observation concerning systolic simplicial complexes.

4.4 Lemma *Let $f : S \to X$ be a simplicial map from a triangulation S of the sphere S^k, for some $k \geq 2$, to a systolic complex X. Then f has a simplicial extension $F : B \to X$, for some triangulation B of the ball B^{k+1} with $\partial B = S$, such that the image $F(B)$ is contained in the full subcomplex of X spanned by the image $f(S)$.*

Proof Denote by K the full subcomplex of X spanned by $f(S)$. Since X is systolic, it is 6-large, and since any full subcomplex of a 6-large complex is 6-large, K is 6-large. In particular, K is aspherical. It follows that the map f, viewed as a continuous map, can be contracted in K to a point. Modifying this contraction to a simplicial map we get an extension F as required. □

We proceed with the proof of Theorem 4.1. Let X be a systolic complex. By Corollary 3.3, it is sufficient to show that the set $X^{(0)}$ of vertices in X, equipped with the metric of polygonal distance in the 1-skeleton of X, is AHA. Fix $r > 0$, a subset $A \subset X^{(0)}$ and consider an arbitrary k-spherical cycle $f : S \to P_r(A)$, for some $k \geq 2$.

4.5 Claim *There is a subdivision S_* of S and a simplicial map $f_* : S_* \to X$ such that*

(1) $f_*(v) = f(v)$ for every vertex v in the initial triangulation S;

(2) for a simplex σ of S, viewed as a subcomplex of S_*, we have diam$(f_*(\sigma)) \leq 4r + 8$.

We skip the proof of the claim until the end of this section (before Subsection 4.6 with comments and questions), and continue the proof of Theorem 4.1, using the claim. Take a simplicial map $f_* : S_* \to X$ as in the claim and put $C = 4r + 8$. For each vertex v in S_* consider its image $f_*(v)$ and denote by $f_0(v)$ a vertex in A at distance not greater than C from $f_*(v)$ in the polygonal distance in $X^{(0)}$. If v is a vertex of the initial triangulation S (before subdividing into S_*), put $f_0(v) := f(v) = f_*(v)$. Choices as above are possible due to conditions (1) and (2) in Claim 4.5. The map f_0 induces the simplicial map $S_* \to P_{2C+1}(A)$ which we also denote f_0. Moreover, maps f_0 and f are clearly simplicially homotopic as maps $S^k \to P_{2C+1}(A)$, in the following sense. There is a triangulation T of the product $S^k \times [0,1]$ and a simplicial map $h : T \to P_{2C+1}(A)$ such that

(h1) triangulation T restricted to $S^k \times \{0\}$ and $S^k \times \{1\}$ is isomorphic to S and S_* respectively;

(h2) restrictions of h to $S^k \times \{0\}$ and $S^k \times \{1\}$ coincide with f and f_0 respectively.

To get the assertion, it is sufficient to get an extension $F_0 : B_0 \to P_{2C+1}(A)$ of f_0, for some triangulation B_0 of the ball B_{k+1} such that $\partial B_0 = S_*$. Indeed, an extension F of f as asserted can be then combined out of F_0 and h.

To get F_0 as above, apply Lemma 4.4 to the map $f_* : S_* \to X$. It gives an extension $F_* : B_* \to X$ with image $F_*(B_*)$ contained in the full subcomplex K in X spanned by the image $f_*(S_*)$. Define F_0 in a similar way as f_0. For every vertex v of B_* choose $F_0(v)$ to be a vertex in A at polygonal distance in $X^{(0)}$ not greater than C. However, for boundary vertices $v \in \partial B_*$ put $F_0(v) := f_0(v)$, which makes sense since $\partial B_* = S_*$. Choices as above are possible due to condition (2) in Claim 4.5 and due to inclusion $F_*(B_*) \subset K$. It is clear from the description that F_0 induces the simplicial map $B_* \to P_{2C+1}(A)$ which is an extension of f_0 as required. Since $2C + 1 = 8r + 17$, the theorem follows.

It remains to prove Claim 4.5.

For every 1-simplex (uw) in S choose a path p_{uw} in the 1-skeleton of X, of length $1 \leq L_{uw} \leq r+2$, connecting the vertices $f(u)$ and $f(w)$. Subdivide (uw) into L_{uw} 1-simplices and define f_* on the subdivided (uw) as its obvious simplicial map on the path p_{uw}.

For every 2-simplex $\sigma = (uwz)$ in S, its boundary $\partial\sigma$ is already subdivided and the map f_* is already defined on $\partial\sigma$. The image $f_*(\partial\sigma)$ is easily seen to be contained in the combinatorial ball $B_{2k}(f(u), X)$, where $k = [r+2]$ is the largest integer not greater than $r+2$. Since balls in systolic complexes are contractible, there is a subdivision σ_* of σ, compatible with the subdivision on $\partial\sigma$, and a simplicial map $f_*^\sigma : \sigma_* \to B_{2n}(f(u), X)$ which extends the already defined map f_* on $\partial\sigma$. Moreover, the diameter of the image $f_*^\sigma(\sigma_*)$ in X is clearly not greater than $4k \leq 4r+8$. Note that in the same way we can define (by induction with respect to dimension) subdivisions σ_* and extensions f_*^σ for all simplices σ in S, with the same estimate for diameters of images.

Composing a subdivision S_* out of the subdivisions σ_* above, and taking f_* to be the union of the maps f_*^σ, we get a map $f_* : S_* \to X$ as required in the claim.

4.6 Comments and questions

1. The arguments in this section give the following more general filling property in systolic complexes: every simplicial map from a simply connected simplicial complex to a systolic complex can be contracted to a point in the simplicial span of its image. In particular, every simplicial cycle modeled on a simply connected space can be filled in the simplicial span of its image by a chain modeled on the cone over this space. The latter can be converted into an asymptotic property slightly stronger than AHA, namely that filling radius for cone fillings of simply connected cycles is uniformly bounded. This property is the strongest presently known to us coarse property of systolic spaces and groups.

2. Systolic groups are, by definition, finitely presented. Are there any finitely presented AHA groups that are not systolic?

3. Are there any AHA groups in dimension ≥ 3 that are not systolic?

5 Filling radius of spherical cycles

To prove the main results of this paper concerning systolic groups, we make a limited use of the fact that these groups are AHA. We only need the S^2FRC property, as introduced below in this section. This property, and the related properties S^kFRC for $k \geq 2$, are easily implied by asymptotic hereditary asphericity.

To define properties S^kFRC we need some preparations. Our exposition uses simplicial chain complexes with arbitrary coefficients.

5.1 Definition A *k-spherical cycle* in a simplicial complex X is a (possibly degenerate) simplicial map $f : S \to X$ from an oriented simplicial k-sphere S. Denote by C_f the simplicial cycle induced by f (the image through f of the fundamental cycle in S). A *filling* of a k-spherical cycle f is a simplicial $(k+1)$-chain D in X such that $\partial D = C_f$. Given a simplicial chain $C = \sum_i t_i\sigma_i$ of dimension n in X, its *support* is the set $\mathrm{supp}(C)$ consisting of all vertices in all underlying simplices $|\sigma_i|$ for which the coefficients t_i are non-zero (here we assume that if σ' is obtained from σ by taking opposite orientation then $\sigma' = -\sigma$, and that the underlying simplices $|\sigma_i|$ of the oriented simplices σ_i occurring in $C = \sum_i t_i\sigma_i$ are distinct for distinct i). The *support* of a spherical cycle $f : S \to X$, denoted $\mathrm{supp}(f)$, is the image through f of the vertex set of S. Note that in general $\mathrm{supp}(f)$ is strictly contained in $\mathrm{supp}(C_f)$.

The *filling radius* of a cycle C in a simplicial complex X is the minimum over all fillings D of the maximal distance of a vertex in $\mathrm{supp}(D)$ from the support of C. Usually, this filling radius grows with the diameter of cycles C. However, it easily follows from Lemma 4.4 that in systolic complexes X the filling radius for k-spherical cycles with $k \geq 2$ is 0 (i.e. each such cycle f has a filling with support contained in $\mathrm{supp}(f)$). This is the model behavior that motivates the next definition, which describes an asymptotic version of this phenomenon.

5.2 Definition A metric space X *has filling radius for k-spherical cycles constant* (shortly, X is S^kFRC), if for every $r > 0$ there is $R \geq r$ such that any k-spherical cycle $f : S \to P_r(X)$ which is null-homologous in $P_r(X)$ has a filling D in $P_R(X)$ with $\mathrm{supp}(D) \subset \mathrm{supp}(f)$.

Note that a more natural definition of S^kFRC, where one asks for the existences of D whose support is distance at most L from the support of f, is in fact equivalent to the one we give. To see this one has to change R to $R + L$, and observe that in the Rips complex $P_{R+L}(X)$ the cycle D is homologous rel boundary to a chain D' whose support is contained in the support of f.

We skip an easy argument, similar to that in the proof of Proposition 3.2, showing that for each k the S^kFRC property is inherited by spaces which uniformly embed into S^kFRC spaces, and hence it is a quasi-isometry invariant. For completeness, we include a proof of the following.

5.3 Lemma *Let X be a metric space which is asymptotically hereditarily aspherical. Then X is S^kFRC for all $k \geq 2$.*

Proof For a fixed $r > 0$ let $R \geq r$ be the corresponding constant for X as in the definition of AHA. We will show that each k-spherical cycle in $P_r(X)$ (regardless if null-homologous in $P_r(X)$ or not) has a desired filling in $P_R(X)$, for the same constant R.

Let $f : S \to P_r(X)$ be any k-spherical cycle. If we denote by $A \subset X$ the set of images through f of all vertices in S, we can view f as a map to the Rips complex $P_r(A)$. By the choice of R, there is a simplicial extension $F : B \to P_R(A)$ of f as in the definition of AHA (Definition 3.1). We can then take as a filling D of f the chain C_F induced by the map F, i.e. the image in $P_R(A)$ of the fundamental chain of B. □

One more straightforward observation concerning S^kFRC property, Proposition 5.4 below, describes its relationship to the *asymptotic dimension* of a metric space. We refer the reader to Gromov [10] or Roe [16] for the definition and some basic facts concerning the latter notion (in [10] this notion is denoted asdim_+ and called *large scale dimension*). For the proof (which is rather straightforward) we refer the reader to Świątkowski [17], where this result occurs as unnumbered proposition in the appendix (p. 220).

5.4 Proposition *If X is a metric space and $\mathrm{asdim} X = p$ then X is S^kFRC for each $k \geq p$.*

5.5 Comments and questions

1. There are many ways in which one can strengthen S^kFRC. One is to replace homological fillings by (simplicial) homotopical ones. Another is to require thin fillings for all spherical cycles, and not only for fillable ones (call this *S^k-thinness*). Also, we can require S^kFRC (or its stronger variants) to hold uniformly in k (at least for $k \geq 2$ or k sufficiently large), i.e. with R not depending on k for any given r. All these strengthening still follow, for all $k \geq 2$, from asymptotic hereditary asphericity.

2. Another way to strengthen S^kFRC is to require constant filling radius for all null-homologous k-dimensional cycles, and not only for spherical ones (call this k-FRC). The same requirement for all cycles, not only for null-homologous ones, is still stronger (call it *k-thinness*). Of course, one can require both conditions above to hold uniformly for appropriate dimensions k. However, k-FRC and k-thinness are not implied by AHA. These conditions are closely related to asymptotic dimension, see 5.5.6.

3. Proofs of Lemma 4.4 and Theorem 4.1, with no changes, show that systolic complexes and groups are k-thin for all cycles that are simply connected (call this SC-k-thin), uniformly in $k \geq 2$ (compare comment 4.6.1 where even stronger, homotopical variant of this property is mentioned). SC-k-thinness is inherited by uniformly embedded subspaces, and in particular by arbitrary finitely generated subgroups. See comment 6.7.2.1 for an interesting application of SC-k-thinness.

4. Note that if X is k-thin for $k \geq 2$ uniformly, then for each $r > 0$ there is $R \geq r$ such that for any $A \subset X$ and any $k \geq 2$ the map $H_k[P_r(A)] \to H_k[P_R(A)]$ has trivial image. This may be viewed as homological analogue of AHA.

5. It follows from the results and arguments in Świątkowski [17, Section 5] that if Γ is a random group as described by M. Gromov in [10, Chapter 9] (and called strongly isoperimetric group in [17]), then Γ is k-thin uniformly for $k \geq 2$. The same is true for groups satisfying various types of small cancellation conditions.

6. By Proposition in the appendix of Świątkowski [17] (p. 220), bounds on the asymptotic dimension of a metric space X imply strengthened variants of S^kFRC. For example, if $\operatorname{asdim} X = p$ then X is p-FRC and it is k-thin for $k \geq p+1$ uniformly. Uniform k-FRC for $k \geq p$ defines what is called in [17, Definition 1.2] the *asymptotic homological dimension*, denoted asdim_h.

7. Define *the asymptotic spherical homological dimension* asdim_h^s as follows: $\operatorname{asdim}_h^s X \leq p$ if and only if X is S^kFRC for all $k \geq p$. We have inequalities $\operatorname{asdim}_h^s \leq \operatorname{asdim}_h \leq \operatorname{asdim}$ (see Świątkowski [17] for the latter inequality). It is also clear that $\operatorname{asdim}_h^s(X) \leq 2$ for each metric space X which is AHA, in particular for systolic complexes and groups. On the other hand, the asymptotic dimensions asdim and asdim_h of a systolic complex or group can be arbitrarily large (see comment 3.5.5).

8. Find a group Γ with $\operatorname{asdim}_h^s \Gamma \leq 2$ which is not AHA. Find a group which is S^kFRC for $k \geq 2$ but not uniformly, or which is not S^k-thin for some $k \geq 2$.

Such a group will not be AHA. Are there other reasons that force a group with $\mathrm{asdim}_h^s\Gamma \leq 2$ to be not AHA? The same problems concern groups Γ with $\mathrm{asdim}_h\Gamma \leq 2$, or even $\mathrm{asdim}\Gamma \leq 2$, compare question 3.5.3.

9. Systolic groups are (in general) not k-thin for any natural k. However, they may have filling radius uniformly bounded for all cycles modeled on a fixed k-dimensional manifold or pseudomanifold M, whatever k and M are. If this is true, the following interesting problem arises. What is the dependence of the uniform filling radius bound on the topology of M? Note that this potentially yields some topological complexity hierarchy in which simply connected manifolds are the least complex ones (compare 5.5.3).

6 Nonpositively curved manifolds

In this section we look at the S^kFRC condition in the context of nonpositively curved manifolds. Our main objective is to show that systolic groups contain no fundamental groups of nonpositively curved manifolds other than surface groups. A crucial observation is the following.

6.1 Proposition *Let X be a simply connected nonpositively curved complete riemannian manifold with* $\dim(X) = n$*. Then X is not S^kFRC for $1 \leq k \leq n-1$.*

Before proving the proposition we derive its consequences. By applying Lemma 5.3 we get the following.

6.2 Corollary *If X is a simply connected nonpositively curved complete riemannian manifold with* $\dim(X) \geq 3$ *then X is not S^2FRC. In particular, X is not AHA.*

To formulate next corollaries we introduce some terminology. A group is called a *nonpositively curved manifold group of dimension n* if it acts isometrically, properly discontinuously and cocompactly on a simply connected nonpositively curved complete riemannian manifold of dimension n (examples are the fundamental groups of closed nonpositively curved manifolds). Since such groups are clearly quasi-isometric to the corresponding manifolds, Corollary 6.2 implies the following.

6.3 Corollary *Nonpositively curved manifold groups of dimension greater than 2 are not S^2FRC. In particular, they are not AHA.*

The above corollary together with Corollary 4.3 imply the following results concerning systolic groups.

6.4 Corollary *A systolic group contains no nonpositively curved manifold group of dimension greater than 2. In particular, it contains no fundamental group of a closed nonpositively curved manifold other than the infinite cyclic group or a surface group (including the torus group $Z \oplus Z$).*

6.5 Corollary *An abelian subgroup of a systolic group has rank at most 2.*

6.6 Corollary *A systolic group contains no product of fundamental groups of closed nonpositively curved manifolds other than $Z \oplus Z$.*

Proof of Proposition 6.1 Fix some k as in the statement and some $r > 0$. For every $R \geq r$ we will construct a k-spherical cycle $f : S \to P_r(X)$ which is null-homologous in $P_r(X)$ but has no filling in $P_R(X)$ with support contained in the the support of f.

Fix a point $p \in X$ and consider the exponential map $E : T_pX \to X$. By the Cartan-Hadamard Theorem, this map is a distance non-decreasing diffeomorphism. Consider the sphere of radius $1000R$ in the tangent space T_pX, centered at 0, and an equatorial sub-sphere S_k of dimension k in it. Denote by N_R the R-neighborhood of S_k in T_pX, i.e. the set consisting of all points in T_pX at distance not greater than R from S_k. Note that the inclusion of S_k in N_R is a homotopy equivalence, and in particular the element $[S_k]$ induced by S_k in the singular homology $H_k(N_R)$ is nontrivial. The same is clearly true for the element $[E(S_k)]$ in $H_k(E(N_R))$.

Fix a triangulation S of the image sphere $E(S_k)$ so that the adjacent vertices are at distance at most r in X. The inclusion of vertices of S in X induces a k-spherical cycle $f : S \to P_r(X)$. Since X is contractible, this cycle is clearly null-homologous in $P_r(X)$. Suppose D is a filling of f in $P_R(X)$ with support contained in the support of f. Denote by U_R the R-neighborhood of the image sphere $E(S_k)$ in X. Since $U_R \subset E(N_R)$ (because the exponential map E is distance non-decreasing), and since $\mathrm{supp}(f) \subset E(S_k)$, it is not hard to make out of D a singular chain in $E(N_R)$ which fills some singular cycle representing the homology class $[E(S_k)]$. This contradicts the fact that $0 \neq [E(S_k)] \in H_k(E(N_R))$, hence the proposition. □

6.7 Comments and questions

1. Is the fundamental group of every closed aspherical manifold of dimension $n \geq 3$ not S^kFRC for some $2 \leq k < n$?

1.1. Note that, due to Theorem 3b.2 (iii) in Davis–Januszkiewicz [6] (and its proof), if X is a closed PL n-manifold with large links then metric spheres in X are homeomorphic to the sphere S^{n-1}. In particular, the proof of Proposition 6.1 above applies to the universal cover of such manifold and, consequently, its fundamental group is not S^kFRC for $1 \leq k \leq n-1$. In this class there are manifolds that (after any smoothing) admit no riemannian metric of nonpositive curvature.

2. Is it true that systolic (or more generally AHA) groups contain no fundamental groups of closed aspherical manifolds of dimension greater than 2? In view of Corollary 3.4 and Lemma 5.3, a positive answer to question 1 above would imply a positive answer to the present question.

2.1. One can answer question 2 above in affirmative, using SC-k-thinness (see comment 5.5.3) rather than AHA, for a class of aspherical manifolds described in Section 5c of [6]. This is some class of negatively curved topological manifolds N with no negatively curved PL metric, existing in any dimension $n \geq 5$. A crucial property of these manifolds is that metric spheres in their universal covers, though not homeomorphic to S^{n-1}, are all simply connected. In particular, fundamental groups of these manifolds N are not SC-$(n-1)$-thin by the argument as in the proof of Proposition 5.1. In view of comment 5.5.3, fundamental groups of these manifolds are not systolic, and cannot occur as subgroups in systolic groups. These groups are different from fundamental groups of riemannian or PL negatively curved manifolds since the ideal boundary is not a manifold.

2.2. Damian Osajda [14] has proved that higher homotopy groups at infinity for systolic complexes are trivial. Since, according to a result of D. Wise [18], finitely presented subgroups of torsion free systolic groups are systolic, this implies that torsion free systolic groups contain no subgroups isomorphic to the fundamental groups of closed aspherical manifolds covered by $\mathbb{R}^n$, for $n \geq 3$.

7 Products

In this section we characterize products of groups that are S^2FRC. Together with the results in the next section, this provides restrictions on product groups that are asymptotically hereditarily aspherical, or that are systolic. The main result is the following.

7.1 Proposition *The product of more than two infinite groups is never S^2FRC. If the product of two infinite groups is S^2FRC then both factor groups are S^1FRC.*

In view of the obvious fact that an infinite group is not S^0FRC, the above proposition is a direct consequence of the following.

7.2 Proposition *Let $k = 0, 1$. If G_1 is a group which is not S^kFRC and G_2 is infinite, then the product $G_1 \times G_2$ is not S^{k+1}FRC.*

Proof We will work with fixed word metrics d_1, d_2 on the groups G_1, G_2 respectively, and with the word metric d on $G_1 \times G_2$ corresponding to the union of generating sets in the factor groups. This metric d can be also described by

$$d((g_1, g_2), (g_1', g_2')) = d_1(g_1, g_1') + d_2(g_2, g_2').$$

Since G_1 is not S^kFRC, there is $r > 0$ such that for every $R \geq r$ there is a k-spherical cycle $f : S \to P_r(G_1)$ which is null-homologous in $P_r(G_1)$ but has no filling in $P_R(G_1)$ with support contained in the support of f. We will construct out of f a $(k+1)$-spherical cycle $f_c : S_c \to P_{r+1}(G_1 \times G_2)$ null-homologous in $P_{r+1}(G_1 \times G_2)$ but having no filling in $P_R(G_1 \times G_2)$ with support contained in the support of f_c. This will show that $G_1 \times G_2$ is not S^{k+1}FRC.

Since a k-spherical cycle $f : S \to P_r(G_1)$ as above is null-homologous in $P_r(G_1)$ it follows that there is a triangulation D of the $(k+1)$-disc, extending the triangulation S viewed as the boundary triangulation of D, and a simplicial map $F : D \to P_r(G_1)$ extending f. Note that here we use the assumption that $k = 0, 1$, since in higher dimensions spheres can bound topologically more complicated chains. For the second factor group G_2, consider the triangulation L of the line segment consisting of $3l + 1$ edges, for some $l \geq R$, with vertices $a_0, a_1, \ldots, a_{3l+1}$ placed in this order on L, and an isometric simplicial embedding $\lambda : L \to P_1(G_2)$, i.e. a simplicial map such that $d_2(\lambda(a_0), \lambda(a_{3l+1})) = 3l + 1$. Such λ clearly exists since the group G_2 is infinite. Take the product $D \times L$, with the product structure of a polyhedral cell complex, and choose a triangulation $(D \times L)^t$ subdividing the cell structure of $D \times L$ without introducing new vertices. The map of the vertex set of $D \times L$ to $G_1 \times G_2$ determined by F and λ induces the simplicial map $F \times \lambda : (D \times L)^t \to P_{r+1}(G_1 \times G_2)$. Since $(D \times L)^t$ is topologically a $(k+2)$-disc, its boundary $\partial(D \times L)^t$ is a triangulation of the $(k+1)$-sphere, and we define the $(k+1)$-spherical cycle f_c to be the restriction

$$f_c = (F \times \lambda)|_{\partial(D \times L)^t} : \partial(D \times L)^t \to P_{r+1}(G_1 \times G_2).$$

Clearly, the map $F \times \lambda$ shows that f_c is null-homologous in $P_{r+1}(G_1 \times G_2)$.

It remains to show that there is no filling of f_c in $P_R(G_1 \times G_2)$ with support contained in the support of f_c.

To any simplicial n-dimensional chain C in $P_R(G_1 \times G_2)$ we associate a simplicial $(n-1)$-dimensional chain $\Delta(C)$ in $P_R(G_1)$, in the following three steps. First, we take C' to be the chain in $P_R(G_1 \times G_2)$ consisting of those n-simplices in C which have at least one vertex not contained in the subcomplex $P_R(G_1 \times B)$, where $B = \{g \in G_2 : d_2(g, \lambda(a_0)) \leq 2l\}$. Next, we take as $\partial_B C'$ the subchain in the boundary $\partial C'$ consisting of those $(n-1)$-simplices that have all vertices in $G_1 \times B$. Finally, we project (the vertices of) $\partial_B C'$ on G_1, getting the chain $\Delta(C)$ in $P_R(G_1)$.

Note that the above description of the operator Δ corresponds, when applied to cycles, to the standard way of defining the homomorphisms

$$H_n(P_R(G_1 \times G_2)) \to H_n(P_R(G_1 \times G_2), P_R(G_1 \times B)) \to$$
$$\to H_{n-1}(P_R(G_1 \times B)) \to H_{n-1}(P_R(G_1))$$

on the level of simplicial cycles. We mention without including further arguments the following easy to verify properties of Δ:

(1) if C is a cycle then $\Delta(C)$ is a cycle,

(2) $\Delta(\partial C) = -\partial(\Delta(C))$ for any chain C,

(3) $\Delta(f_c) = f$.

Now, suppose that D is a filling of f_c in $P_R(G_1 \times G_2)$ with support contained in $\mathrm{supp}(f_c)$. Then, by the above properties of Δ, the chain $\Delta(D)$ is (up to sign) a filling of f in $P_R(G_1)$. We check the properties of the support of $\Delta(D)$. Note first that all vertices in the simplices of D' lie at distance greater than R from $G_1 \times \{\lambda(a_0)\}$. Consequently, the vertices of $\partial_B D'$ lie at distances greater than R from $G_1 \times \{\lambda(a_0), \lambda(a_{3l+1})\}$. Since, denoting by $L^{(0)}$ the set of vertices in L, we have

$$\mathrm{supp}(f_c) \subset G_1 \times \{\lambda(a_0), \lambda(a_{3l+1})\} \cup \mathrm{supp}(f) \times \lambda(L^{(0)})$$

and

$$\mathrm{supp}(\partial_B D') \subset \mathrm{supp}(D) \subset \mathrm{supp}(f_c),$$

it follows that $\mathrm{supp}(\partial_B D') \subset \mathrm{supp}(f) \times \lambda(L^{(0)})$. After projecting to G_1, we clearly get $\mathrm{supp}(\Delta(D)) \subset \mathrm{supp}(f)$, a contradiction. $\square$

7.3 Comments and questions

1. A method used in the proof of Proposition 7.2 allows to show that, for any k, the product of an infinite group with a group which is not k-FRC is not $(k+1)$-FRC (see 5.5.2 for the definition of k-FRC). Is the statement of Proposition 7.2 still true for $k > 1$?

2. Is it true in general that the product of a group which is not k-FRC with a group which is not l-FRC is always not $(k+l)$-FRC? Is the product of a group which is not S^kFRC with a group which is not S^lFRC always not S^{k+l}FRC?

3. Is the product of any two S^1FRC groups S^2FRC (compare 8.9.3)? Is the same true for products of S^kFRC or k-FRC groups for arbitrary k?

8 S^1FRC groups

In this section we establish several properties of S^1FRC groups. We use them, together with the results from the previous section, to get restrictions on product groups that are AHA or systolic.

Our main results are the following.

8.1 Proposition *If a finitely presented group is S^1FRC then it is hyperbolic.*

8.2 Proposition *The topological dimension of the Gromov boundary of an S^1FRC hyperbolic group is zero.*

In view of the well known fact that a hyperbolic group with zero-dimensional boundary is virtually free, the above propositions imply the following.

8.3 Corollary *Any finitely presented S^1FRC group is virtually free.*

Before proving Propositions 8.1 and 8.2 we derive some of their consequences for S^2FRC, systolic and AHA groups.

8.4 Corollary *The product of two infinite finitely presented groups is S^2FRC if and only if both factors are virtually free.*

Proof One implication follows from the fact that asymptotic dimension of the product of two finitely generated free groups is 2, and from Proposition 5.4. The other implication follows from Proposition 7.1 and Corollary 8.3. □

8.5 Corollary *If the product of two infinite groups is systolic then both factor groups are virtually free.*

Proof Since the product group acts simplicially properly discontinuously and cocompactly on a simplicial complex, any systolic group is finitely presented. If it is a product then all factors are finitely presented. The rest of the argument is then as in the proof of Corollary 8.4. □

Remark The converse to Corollary 8.5 holds true. Tomasz Elsner and Piotr Przytycki pointed out to us how to construct systolic complexes with a geometric action of the product of two finitely generated virtually free groups. Surprisingly, the dimension of the complexes is 3 and not 2 as one would expect.

Since AHA groups are S^2FRC (Lemma 5.3), we also have the following.

8.6 Corollary *If the product of two infinite finitely presented groups is AHA then both factors are virtually free.*

Note that Proposition 7.1 alone, without using of results from this section, implies the following result that complements Corollaries 8.5 and 8.6.

8.7 Corollary *The product of more than two infinite groups is never AHA and, in particular, never systolic.*

Proof of Proposition 8.1 Let Γ be any finitely presented group which is S^1FRC. Fix a finite generating set Σ and the corresponding word metric d_Σ in Γ. Since Γ is finitely presented, there is $r \geq 1$ such that each 1-spherical cycle in the Rips' complex $P_1(\Gamma)$ is null-homologous in $P_r(\Gamma)$. For example, we can take as r the maximal length of a relation from some finite presentation for Γ with Σ as the generating set. Let $R \geq r$ be a constant occurring in the definition of S^1FRC property (Definition 5.2), i.e. such that each 1-spherical cycle $f : S \to P_r(\Gamma)$ which is null-homologous in $P_r(\Gamma)$, has a filling D in $P_R(\Gamma)$ with support contained in the support of f. We will show that Γ is δ-hyperbolic for $\delta = 100R$. The constant δ we obtain is obviously far from optimal.

Let T be a geodesic triangle in the Cayley graph $C(\Gamma, \Sigma)$, with vertices in Γ. View T as a 1-spherical cycle in $P_1(\Gamma)$, and denote the corresponding simplicial map from a 1-sphere by $f_T : S_T \to P_1(\Gamma)$. Since f_T has a filling in $P_r(\Gamma)$ (due to finite presentability, as above), there is a filling D of f_T in $P_R(\Gamma)$ such that $\mathrm{supp}(D) \subset \mathrm{supp}(f_T)$. This implies the following.

Claim 1. *The map f_T (viewed as a continuous map) is a loop which is contractible (i.e. homotopic to a constant map) in the Rips complex $P_R[supp(f_T)]$.*

To prove δ-hyperbolicity of Γ, for $\delta = 100R$, we need to show that, for any side A of the triangle T and for every vertex x on A, there is a vertex y in the union $B \cup C$ of two other sides such that $d_\Sigma(x, y) \leq 100R$. Note that if length of A is not greater than $200R$ then this condition is trivially satisfied. If length of A is greater than $200R$, choose vertices $p_0, p_1, \ldots, p_m$ on A, occurring in this order, with p_0, p_m the endpoints of A, such that for $i = 0, 1, \ldots, m-1$ we have

$$50R < d_\Sigma(p_i, p_{i+1}) \leq 98R.$$

This clearly can be done, for example by taking all distances minimal (i.e. as close from above to $50R$ as possible), except the last two possibly bigger. For each $i = 0, 1, \ldots, m$ put $U_i := \{g \in \Gamma : d_\Sigma(g, p_i) \leq 50R\}$. We then have:

(a1) $U_i \cap U_{i+1} \neq \emptyset$ for $i = 0, 1, \ldots, m-1$,

(a2) $U_i \cap U_j = \emptyset$ for $|i - j| > 1$,

(a3) $A^{(0)} \subset \cup_{i=0}^m U_i$, where $A^{(0)}$

is the set of vertices on the side A.

We omit the straightforward check of these properties.

For every vertex $v \in B^{(0)} \cup C^{(0)}$ put $U_v := \{g \in \Gamma : d_\Sigma(g, v) \leq R\}$. Denote by $\mathcal{U}$ the family consisting of the sets U_i for $i = 0, 1, \ldots, m$ and the sets U_v for all $v \in B^{(0)} \cup C^{(0)}$. It follows from property (a3) that

(a4) $\text{supp}(f_T) \subset \cup\mathcal{U}$.

To finish the proof, we will show the following.

Claim 2. *For each $i = 0, 1, \ldots, m-1$ there is $v \in B^{(0)} \cup C^{(0)}$ such that $U_i \cap U_{i+1} \cap U_v \neq \emptyset$.*

Given the Claim 2, one concludes the proof of Proposition 8.1 as follows. For each $x \in A^{(0)}$ there is i such that $d_\Sigma(x, p_i) \leq 49R$. By Claim 2, there is $v \in B^{(0)} \cup C^{(0)}$ with $d_\Sigma(p_i, v) \leq 50R + R = 51R$. Consequently, $d_\Sigma(x, v) \leq 100R$, and hence Γ is δ-hyperbolic with $\delta = 100R$.

It remains to prove Claim 2. To do this, denote by $N(\mathcal{U})$ the *nerve* of the family $\mathcal{U}$, i.e. a simplicial complex whose vertices are the sets from $\mathcal{U}$ and whose simplices correspond to subfamilies of $\mathcal{U}$ that have nonempty intersection. For each vertex $w \in \mathrm{supp}(f_T)$ choose a set from $\mathcal{U}$ containing w, denoting it by $h(w)$, as follows. If $w \in A^{(0)}$ choose as $h(w)$ one of the sets U_i and if $w \in B^{(0)} \cup C^{(0)} \setminus \{p_0, p_m\}$ put $h(w) = U_w$. Note that if vertices $w_1, w_2 \in \mathrm{supp}(f_T)$ are adjacent in the Rips' complex $P_R\Gamma$ then the corresponding sets $h(w_1), h(w_2)$ have nonempty intersection. Consequently, h induces a simplicial map $H : P_R[\mathrm{supp}(f_T)] \to N(\mathcal{U})$.

Suppose, contrary to Claim 2, that there is i such that $U_i \cap U_{i+1} \cap U_v = \emptyset$ for each $v \in B^{(0)} \cup C^{(0)}$. Together with property (a2) above this implies that the edge (U_i, U_{i+1}) in the nerve $N(\mathcal{U})$ is isolated, i.e. not contained in any 2-simplex. We claim that then the composed map $H \circ f_T : S_T \to N(\mathcal{U})$ is not contractible. To see this, note that $h(p_i) = U_i$, $h(p_{i+1}) = U_{i+1}$ and $h(w) \in \{U_i, U_{i+1}\}$ for all vertices w contained in A lying between p_i and p_{i+1}. This means that part of the 1-sphere S_T corresponding to the segment $[p_i, p_{i+1}] \subset A$ is mapped through H on the edge (U_i, U_{i+1}). Moreover, by definition of h, no edge in S_T not contained in the above segment $[p_i, p_{i+1}]$ is mapped on the edge (U_i, U_{i+1}) through H. As a consequence, the loop $H \circ f_T : S_T \to N(\mathcal{U})$ passes through the edge (U_i, U_{i+1}) an odd number of times, and since the edge is isolated, the loop is not contractible. Since this contradicts the assertion of Claim 1, the proof of Proposition 8.1 is completed. □

Proof of Proposition 8.2 We will prove that if G is a hyperbolic group such that $\dim \partial_\infty G \geq 1$ then G is not S^1FRC.

Given $\epsilon > 0$, an *ϵ-path* in a metric space (X, d) connecting points $a, b \in X$ is a finite sequence $x_0, x_1, \dots, x_n$ of points in X such that $x_0 = a$, $x_n = b$, and $d(x_{i-1}, x_i) < \epsilon$ for $i = 1, \dots, n$. We start with the following auxiliary fact, certainly well known, including its proof for completeness.

8.8 Fact *Let (X, d) be a compact metric space with $\dim X \geq 1$. Then there are distinct points $a, b \in X$ such that for every $\epsilon > 0$, a, b can be connected by an ϵ-path in X .*

Proof We will show that if points a, b as in the assertion do not exist then $\dim X = 0$. More precisely, we will show that every point $a \in X$ has arbitrarily small neighborhoods that are open and closed.

Let U be any open neighborhood of a in X. Consider the real function $f : X \setminus \{a\} \to \mathbb{R}$ defined by

$$f(x) := \inf\{\epsilon > 0 : \ x \text{ is connected with } a \text{ by an } \epsilon\text{-path in } X\}.$$

By our assumption in the proof, the function f is positive at every point. Moreover, it is easily seen to be locally constant, hence continuous. Since the complement $X \setminus U$ is compact, f has minimum $m > 0$ at this complement. Take any $0 < \delta < m$ and put

$$V := \{x \in X : \ x \text{ can be connected with } a \text{ by a } \delta\text{-path in } X\}.$$

It is straightforward that V is open and closed, and that $a \in V \subset U$. This completes the proof of Fact 7.8. □

Coming back to the proof of Proposition 8.2, note that Gromov boundary $\partial_\infty G$ is a compact metric space. Existence of ϵ-paths as in Fact 8.8 between some points $a, b \in \partial_\infty G$ allows to construct S^1-cycles $f : S \to P_r G$, for some fixed $r > 0$, with arbitrarily large filling radii, using a method from Świątkowski [17]. More precisely, if $\dim \partial_\infty G \geq 1$, there is a proper compact subset $K \subset \partial_\infty G$ with $\dim K \geq 1$, and distinct points $a, b \in K$ which can be connected by an ϵ-path c_ϵ in K for every $\epsilon > 0$. Fixing point $s \in \partial_\infty G \backslash K$, it is possible to project paths c_ϵ to various horospheres in G centered at s, by the projections described in Section 2 of [17]. If ϵ is small enough, we can construct an S^1-cycle by taking projections of c_ϵ on two sufficiently distant horospheres, and connecting them with two segments contained in projection rays corresponding to points a, b (the endpoints of c_ϵ). The fact that S^1-cycles in some $P_r G$ constructed in this way have arbitrarily large filling radii follows from the argument as in the proof of Theorem 3.1 in [17]. The argument is the same as in the proof of Theorem 4.1 in [17] and we omit further details. □

8.9 Comments and questions

1. The results of this section motivate the following general question. Which products of infinite groups are "2-dimensional"? "Dimension 2" in this question can be meant in any of the following senses: S^2FRC, AHA, systolic, asymptotically or (virtually) cohomologically 2-dimensional, S^2-thin, 2-FRC, 2-thin, etc. Here are some observations and more detailed questions concerning this subject.

1.1. Note that, since the product of two virtually free groups is asymptotically 2-dimensional (as the product of two asymptotically 1-dimensional factors), AHA (comment 3.5.2.2), and S^2-thin, and since S^2-FRC is implied by any of these three conditions (Proposition 5.4), Corollary 8.4 can be strengthened as follows: the product of two infinite finitely presented groups is asymptotically 2-dimensional (or AHA or S^2-thin) if and only if both factor groups are virtually free.

1.2. It was observed by Tomasz Elsner and Piotr Przytycki that product of two finitely generated virtually free groups is systolic.

1.3. Note that the product of two groups of asymptotic dimension 1 has asymptotic dimension at most 2 (see Roe [16, Proposition 9.11]) and hence, due to Proposition 5.4, is S^2FRC. Is it AHA? Are there S^1FRC groups that are not asymptotically 1-dimensional? If the answer to the last question is negative then product of any two S^1FRC groups is S^2FRC.

2. Which extensions of infinite groups by infinite groups are systolic, S^2FRC or AHA? Note that in the systolic case this class probably coincides with the class of systolic products, because systolic groups are biautomatic, see Januszkiewicz–Świątkowski [12]. (Biautomatic groups satisfy quadratic isoperimetric inequality, their solvable subgroups are virtually abelian and their infinite cyclic subgroups are undistorted. This probably excludes most of non-product extensions.) On the other hand, Baumslag-Solitar groups are AHA as they act on the product of a tree and the line (compare 3.5.2.1).

3. What are the relationships between the following classes of "1-dimensional" groups: virtually free, asymptotic dimension 1, S^1-thin, S^1-FRC? In the finitely presented case all these classes coincide, and in the infinitely presented one there are obvious inclusions. A similar question can be asked for various classes of "2-dimensional" and "higher dimensional" groups.

4. Finite presentability assumption in Corollary 8.3 is essential, see Fujiwara–Whyte [8] and Nowak [13].

9 Second isoperimetric inequality

In this section we show that the isoperimetric function for 2-spherical cycles (the so called second isoperimetric function) is linear in systolic complexes. Our exposition uses simplicial chain complexes with integer coefficients.

To formulate the main result of this section we need the following.

9.1 Definition Let $C = \sum_i t_i\sigma_i$ be an n-chain in a simplicial complex X. Suppose (without loss of generality) that the underlying simplices $|\sigma_i|$ for which $t_i \neq 0$ are distinct for distinct i. The (n-dimensional) *volume* of C, $V_n(C)$, is defined as $V_n(C) := \sum_i |t_i|$. The (2-dimensional) volume of a spherical 2-cycle $f : S \to X$, $V_2(f)$, is the number of 2-simplices in S that are mapped by f

on 2-simplices (i.e. the number of 2-simplices on which f is injective). Note that if C_f is the chain induced by f $V_2(C_f) \leq V_2(f)$, and the inequality can be strict.

9.2 Theorem *Given any spherical 2-cycle $f : S \to X$ in a systolic simplicial complex X, there exists a filling D of f such that*

$$V_3(D) \leq \frac{3}{2} \cdot V_2(f).$$

Proof The argument is an induction on the number of 2-simplices in the triangulation S of the 2-sphere.

The smallest triangulation S of the 2-sphere consists of four 2-simplices (the boundary triangulation of a 3-simplex). Then any nontrivial (i.e. such that $[f] \neq 0$) spherical 2-cycle $f : S \to X$ has volume $V_2(f) = 4$. Moreover, by the fact that X is flag, there is a filling D of f consisting of a single simplex, i.e. such that $V_3(D) = 1$. The assertion of the theorem is clearly satisfied in this case.

Now let S be any triangulation of the 2-sphere, and let $f : S \to X$ be any simplicial map. We consider three cases. The first two deal with rather degenerate situations, making no use of the fact that X is systolic. The key systolic argument occurs in the third (most regular) case.

Case 1: *f is degenerate.*

Consider a 1-simplex $e = (v, w)$ in S for which $f(v) = f(w)$, and let $\tau_1 = (v, w, u_1)$, $\tau_2 = (v, w, u_2)$ be the 2-simplices adjacent to e in S. Delete from S the interior of the union $\tau_1 \cup \tau_2$ and glue the obtained boundary edges in pairs: (u_1, v) with (u_1, w) and (u_2, v) with (u_2, w). As a result we obtain a multisimplicial triangulation S' of the 2-sphere, which may contain potentially a double edge or a double triangle (i.e two 2-simplices that share boundaries), but never a loop. Suppose first that S' is simplicial. The simplicial map $f : S \to X$ induces the simplicial map $f' : S' \to X$ in the obvious way. Moreover, we have the equality $[f'] = [f]$ for the induced cycles, and thus every filling of f' is also a filling of f. Consequently, since S' consists of less 2-simplices than S, the assertion follows for f by induction.

It remains to deal with the case when S' is not a simplicial complex. This case splits into two subcases. First is when S' consists of two 2-simplices that share their boundaries. But then $[f'] = 0$, hence $[f] = 0$, and the assertion holds trivially. Otherwise, there must be a vertex y in S such that (v, y) and

(w, y) are both the 1-simplices in S. Since by passing from S to S' the vertices v and w are identified, this leads to a double edge in S'. We still denote by (w, y) and (v, y) the corresponding edges in S'. Let y_i be all the vertices in S satisfying the property as y above. Denote by $e_i = (v, y_i)$ and $e'_i = (w, y_i)$ the corresponding edges in S'. The union of all these edges splits S' into open discs D_j, and the boundary of each D_j is the union of edges from some pair e_i, e'_i or from two such pairs. In any case, we identify the edges from the pairs e_i, e'_i in the closure of each D_j, getting 2-spheres S_j of the following two kinds. Some of these spheres may consist of two 2-simplices identified along whole boundaries. It is not hard to see that all other spheres S_j are the honest simplicial spheres. We will denote them as S_{j_k}. Clearly, the map $f : S \to X$ induces simplicial maps $f_{j_k} : S_{j_k} \to X$ and we have $\sum_k [f_{j_k}] = [f]$. It follows that the sum $\sum_k D_{j_k}$ of any fillings D_{j_k} of f_{j_k} is a filling of f. Since moreover $\sum_k V_2(f_{j_k}) \leq V_2(f) - 2$, the assertion follows for f by induction.

Case 2: *f is non-degenerate and the images of some two adjacent 2-simplices in S coincide.*

Denote by $\tau_1 = (v, w, u_1)$ and $\tau_2 = (v, w, u_2)$ the two adjacent simplices in S whose images through f coincide. Remove from S the interior of the union $\tau_1 \cup \tau_2$, and glue the edges of the resulting boundary in pairs: (u_1, v) with (u_2, v) and (u_1, w) with (u_2, w). After gluing we obtain a cell decomposition S' of the 2-sphere. We omit the easy case when S' is a simplicial triangulation and pass directly to the opposite case.

The only reason for S' not to be simplicial is that there is an edge $e = (u_1, u_2)$ in S (which becomes a loop in S', still denoted e) or that there are pairs of edges $e_i = (u_1, z_i)$ and $e'_i = (u_2, z_i)$ in S (which become the double edges in S', still denoted e_i, e'_i). Split S' into open discs D_j along the union of all edges e_i, e'_i and e as above. Consider those discs D_{j_k} in the splitting that contain more than two open 2-simplices of S'. Observe that the loop e (if exists) is not contained in the boundary of any D_{j_k}. Moreover, the boundary of each D_{j_k} is the union of edges from some pair e_i, e'_i or from two such pairs. We identify the edges from the pairs e_i, e'_i in the closure of each D_{j_k}, getting 2-spheres S_{j_k}. It is not hard to see that each S_{j_k} is simplicial. Note that the map f induces in the obvious way the simplicial maps $f_{j_k} : S_{j_k} \to X$. The rest of the argument is as in Case 1.

Case 3: *f is non-degenerate and no two adjacent 2-simplices from S are mapped to the same simplex.*

By the Euler characteristic argument (or combinatorial Gauss-Bonnet theorem), there exists a vertex v in S for which the link S_v is a polygonal cycle consisting of $m < 6$ edges. Since the map f is non-degenerate, it induces the non-degenerate simplicial map $f_v : S_v \to X_{f(v)}$ on links. Since no two adjacent 2-simplices of S have the same image, the map f_v is locally injective, i.e. no two adjacent edges in S_v are mapped to the same edge. Since length m of S_v is at most 5, the map f_v is in fact injective. Its image in the link $X_{f(v)}$ is a cycle of length m. Since X is systolic, the link $X_{f(v)}$ is 6-large, and thus the cycle $f_v(S_v)$ has a disc filling Δ_1 in $X_{f(v)}$ consisting of $m-2$ simplices of dimension 2. Viewing the link $X_{f(v)}$ as a subcomplex in X, the cycle $f_v(S_v)$ has also the disc filling Δ_2 in X equal to the join of the cycle with v. Moreover, there is a 3-chain C in X, consisting of $m-2$ simplices of dimension 3, such that $\Delta_2 - \Delta_1 = \partial C$ and the support of C consists of v and the vertices in $f_v(S_v)$.

We now modify the spherical 2-cycle $f : S \to X$ by removing from S the open star of the vertex v, and by glueing to the obtained boundary S_v the disc Δ made of $m-2$ simplices of dimension 2 in the way compatible with the filling Δ_1 of $f_v(S_v)$. If the obtained 2-sphere S' is simplicial, we define the simplicial map $f' : S' \to X$ by taking the compatibility map $\Delta \to \Delta_1$ as the restriction of f' to Δ, and by putting $f' = f$ outside Δ. Note that $V_2(f') = V_2(f) - 2$ and the difference $[f] - [f']$ is equal to $\Delta_2 - \Delta_1$. Let D be a filling of f' satisfying the assertion of the theorem (which exists due to inductive assumption). By what was said above about the chain C, the sum $D + C$ is a filling of f with $V_3(D+C) \leq V_3(D) + m - 2$. Since $m - 2 \leq 3$, the chain $D + C$ is a filling of f as required.

It remains to consider the subcase in which the obtained 2-sphere S' is not simplicial. The only reason for S' not to be simplicial is as follows. For an edge δ of the inserted disc Δ not contained in the boundary $\partial\Delta$, its endpoints (viewed as vertices in S_v) may be connected by an edge e in S. In S' we then get two edges δ and e connecting the same pair of vertices. Let (δ_i, e_i) be all pairs of double edges in S' as above. Note that for distinct i the corresponding edges δ_i are distinct, and hence there are at most $m-3$ such pairs. The union of all edges δ_i and e_i from these pairs splits S' into open discs D_j, and the boundary of each D_j is the union of edges from some pair (δ_i, e_i) or from two such pairs. Similarly as in Case 1, we identify the edges from the pairs (δ_i, e_i) in the closure of each D_j, getting 2-spheres S_j. The rest of the argument is as at the end of Case 1, combined with the estimates above in this case (for S' simplicial). We omit the obvious details. □

9.3 Comments and questions

1. One is tempted to investigate higher dimensional isoperimetric inequalities for systolic complexes, similar to that in Theorem 9.2. It seems that proving

them requires techniques significantly different from ours. On the other hand, tangible meaning and applications of such results remains unclear to us at this stage.

2. What are the possibilities for the second isoperimetric function for AHA complexes and groups?

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