Ideal boundary of 7-systolic complexes and groups

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Abstract

We prove that ideal boundary of a 7-systolic group is strongly hereditarily aspherical. For some class of 7-systolic groups we show their boundaries are connected and without local cut points, thus getting some results concerning splittings of those groups.

1 Introduction

The notion of k-systolic ($k \geq 6$ being a natural number) complexes was introduced by T. Januszkiewicz and J. Świątkowski [JS1] and, independently, by F. Haglund [H] as combinatorial analogue of nonpositively curved spaces. Those complexes are simply connected simplicial complexes satisfying some combinatorial local conditions. Roughly speaking there is a lower bound for the length of "essential" closed paths in a one-skeleton of every link.

A group acting geometrically by automorphisms on a k-systolic complex is called a k-systolic group. Examples of such torsion free groups of arbitrary large cohomological dimension are constructed by Januszkiewicz and Świątkowski [JS1], for every $k \geq 6$. Those examples are fundamental groups of some simplices of groups. In the same paper it is proved that 7-systolic groups are Gromov-hyperbolic.

In this paper we study 7-systolic complexes and groups and in particular their ideal boundaries. Our main result is the following.

Theorem 1 (Theorem 4.2 in Section 4) *The ideal boundary of a 7-systolic group is a strongly hereditarily aspherical compactum.*

The notion of the strong hereditary asphericity (see Section 2.3 for the precise definition) was introduced by R. J. Daverman [Da]. Roughly speaking a space is hereditarily aspherical if each of its closed subsets is aspherical. The significance of this notion follows from the fact that a cell-like map defined on a strongly hereditarily aspherical compactum does not raise dimension.

Theorem 1 shows that 7-systolic groups are quite different from many classical hyperbolic groups. It gives also new examples of topological spaces that can occur as boundaries of hyperbolic groups. The question about the spaces being such boundaries is well understood only in dimensions 0 and 1—compare Kapovich and Kleiner [KaKl]. It is still not known which topological spaces can be higher dimensional boundaries of hyperbolic groups and only few homeomorphism types of such spaces are known—see Benakli and Kapovich [BeK] and discussion in p. 1) of Remarks in Section 4. Moreover, we show (see below) that for certain classes of complexes (and groups) their ideal boundaries are "simple" in a sense—they are connected and have no local cut points.

In order to prove the main theorem we define (in Section 3) an inverse system of combinatorial spheres in a 7-systolic complex and projections onto them, whose inverse limit is the ideal boundary of the complex. It should be noticed that even in the more general systolic (which means 6-systolic) case some inverse system like that can be defined (cf. Januszkiewicz and Świątkowski [JS1, Section 8]). However its properties do not allow to consider it as a right tool to define a reasonable boundary of a systolic complex or group. In particular the inverse limit of this standard systolic inverse system in a 7-systolic case is not the Gromov boundary. On the other hand, our construction is not valid in general systolic case, although some results are probably true for hyperbolic systolic (not necessarily 7-systolic) groups.

In Section 5 we study further properties of boundaries of some 7-systolic complexes. In particular we prove the following theorem, which is a special case of Theorem 5.6 proved in that section.

Theorem 2 (Corollary 5.7 in Section 5) *Let X be a locally finite 7-systolic normal pseudomanifold of dimension at least 3. Then its ideal boundary ∂X is connected and has no local cut points.*

Via the results of Stallings [St] and the ones of Bowditch [Bow] the latter theorem implies the following.

Theorem 3 (Corollary 5.9 in Section 5) *A group acting geometrically by automorphisms on a locally finite 7-systolic normal pseudomanifold of dimension at least 3 does not split essentially, as an amalgamated product or as an HNN-extension, over a finite nor two-ended group.*

Groups acting geometrically on such pseudomanifolds of arbitrary large dimension were constructed in Januszkiewicz and Świątkowski [JS1] and are the only 7-systolic groups of cohomological dimension greater than 2 known to us at the moment.

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2 Preliminaries

2.1 Simplicial complexes

In this section we recall some definitions and fix the notation.

Let X be a simplicial complex. We denote by X' the first barycentric subdivision of X. For a natural number k, we denote by $X^{(k)}$ the *k-skeleton* of X, i.e. the union of all simplices of X, of dimension at most k. For a given subset $C = \{v_1, v_2, ..., v_l\}$ of $X^{(0)}$ we denote by $< v_1, v_2, ..., v_l >$ the minimal simplex in X containing C—the simplex *spanned* by C. We denote by X_σ the link of a given simplex σ in X. A simplicial complex X is *flag* if every set B of pairwise connected (by edges) vertices of X spans a simplex in X.

Recall that a subcomplex Y of X is *full* if every set B of vertices of Y spanning a simplex of X spans a simplex in Y. We denote by $\sigma * \rho$ the join of simplices σ and ρ.

If it is not stated otherwise a simplicial complex X is equipped with a path metric d_X for which every k-simplex of X is isometric to the regular Euclidean k-simplex.

A simplicial complex X is called a *chamber complex of dimension* n if it is the union of n-simplices (which are called *chambers* of X) and for every $(n-1)$-dimensional face of X there exist at least two chambers containing that face. It is easy to see that links in a chamber complex are themselves chamber complexes. A *gallery* in a chamber complex is a finite sequence of maximal simplices such that two consecutive simplices share a common face of codimension 1. A chamber complex is said to be *gallery connected* if any two chambers can be connected by a gallery. Chamber complex is *normal* if it is gallery connected and all its links of dimension above 0 are gallery connected. A chamber complex is a *pseudomanifold* if every codimension one face belongs to exactly two chambers.

2.2 Systolic complexes and groups

We follow here Januszkiewicz and Świątkowski [JS1], [JS2]. For a given natural number $k \geq 4$, a simplicial complex X is *k-large* if it is flag and every cycle γ in X (i.e. a subcomplex homeomorphic to the circle) of length $4 \leq |\gamma| < k$ has a diagonal (i.e. an edge connecting two nonconsecutive vertices in γ). Here $|\gamma|$ denotes the number of edges of γ.

A simplicial complex X is *locally k-large* if for every simplex $\sigma \neq \emptyset$ of X its link X_σ in X is k-large.

X is *k-systolic* if it is locally k-large, connected and simply connected.

Because $k = 6$ is of special importance in that theory, 6-systolic complexes are called *systolic*.

A group acting geometrically (i.e. properly discontinuously and cocompactly) by simplicial automorphisms on a k-systolic (resp. systolic) complex is called *k-systolic* (resp. *systolic*). Free groups and fundamental groups of surfaces are systolic groups. In Januszkiewicz and Świątkowski [JS1], for arbitrary k and n, a torsion free k-systolic group of cohomological dimension n is constructed. Those groups are the fundamental groups of some simplices of groups.

In the rest of this subsection we list some results concerning systolic complexes and groups. We begin with the easy facts whose proofs can be found in [JS1].

Proposition 2.1 *1. If $k > m$ and X is k-large then X is also m-large.*

2. A full subcomplex in a (locally) k-large complex is (locally) k-large.

3. Links of a k-large complex are k-large.

4. There is no k-large triangulation of the 2-sphere for $k \geq 6$. Hence no triangulation of a manifold of dimension above 2 is 6-large since 2-spheres occur as links of some simplices in that case.

The following property of 7-systolic complexes is crucial for this paper.

Theorem 2.2 *[JS1, Theorem 2.1] Let X be a 7-systolic complex. Then the 1-skeleton of X with its standard geodesic metric is δ-hyperbolic with $\delta = 2\frac{1}{2}$.*

Thus 7-systolic groups are word-hyperbolic.

In [JS1, Sections 3 and 7] the notion of a *convex* subcomplex of a systolic complex is introduced. A simplex is a convex subcomplex.

For a simplicial complex X and its subcomplex Q we can define a *closed combinatorial ball of radius i around Q in X*, $B_i(Q, X)$, inductively: $B_0(Q, X) = Q$ and $B_i(Q, X) = \bigcup \{\tau : \tau \cap B_{i-1}(Q, X) \neq \emptyset\}$, for every positive natural number i.

By $S_i(Q, X)$ we denote the subcomplex of $B_i(Q, X)$ spanned by the vertices at combinatorial distance i from Q, i.e. not belonging to $B_{i-1}(Q, X)$. By $\overset{\circ}{B}_i(Q, X)$ we denote the *interior* of the closed combinatorial i-ball around σ in X, i.e. $\overset{\circ}{B}_i(Q, X) = B_i(Q, X) \setminus S_i(Q, X)$.

For the rest of this section let X denote a systolic complex and Q its convex subcomplex.

Lemma 2.3 *[JS1, Lemmas 7.5 and 7.6] The sphere $S_i(Q, X)$ and the ball $B_i(Q, X)$ are full subcomplexes of X and they are k-large.*

Lemma 2.4 *[JS1, Sect. 7] If $i > 0$, then for every simplex $\tau \in S_i(Q, X)$, $\rho = \partial B_{i-1}(Q, X) \cap X_\tau$ is a single simplex and $X_\tau \cap B_i(Q, X) = B_1(\rho, X_\tau)$ and $X_\tau \cap S_i(Q, X) = S_1(\rho, X_\tau)$.*

In the rest of the paper we call the simplex ρ, as in the above lemma the *projection of* τ *on* $S_{i-1}(Q,X)$.

The universal cover of a connected locally 6-large simplicial complex is systolic and the folowing fact holds.

Theorem 2.5 *[JS1, Theorem 4.1] The universal cover of a finite dimensional connected locally* 6*-large simplicial complex is contractible.*

The proof of this theorem uses the projections onto closed combinatorial balls (compare [JS1, Section 8]). Restrictions of those projections to spheres

$$\pi_{B_i(Q,X)}|_{S_{i+1}(Q,X)}\colon S_{i+1}(Q,X) \to S_i(Q,X)$$

have some properties that do not allow us to use them in order to define a reasonable boundary of a systolic complex. Thus in Section 3 we define, only for 7-systolic complexes, other maps between spheres.

Lemma 2.6 *Let* $k \geq 6$*, let* Y *be a* k*-large simplicial complex and let* σ *be a simplex of* X*. If* $p : X \to Y$ *is the universal cover of* Y *and* $m < \frac{k-1}{2}$ *then for* $i = 0, 1, 2, ..., m$ *the map* $p|_{B_i(\sigma,X)} : B_i(\sigma, X) \to p(B_i(\sigma, X))$ *is an isomorphism.*

Here we recall two results concerning systolic chamber complexes.

Lemma 2.7 *[O, Lemma 4.1] Let* X *be a systolic chamber complex of dimension* $n \geq 1$ *and* τ *its simplex. Then* $S_k(\tau, X)$ *is an* $(n-1)$*-dimensional chamber complex, for every* $k \geq 1$*.*

Lemma 2.8 *[O, Lemma 4.2] Let* X *be a systolic chamber complex of dimension* $n \geq 1$ *and* τ *its simplex. Let* σ *be an* $(n-1)$*-dimensional simplex of* S_k*. Then there exists a vertex* v *at a distance* $k+1$ *from* τ *such that* $v * \sigma$ *is a simplex of* X*.*

2.3 Strongly hereditarily aspherical compacta

The notion of strongly hereditarily aspherical compacta was introduced by R. J. Daverman [Da]. A compact metric space Z is *strongly hereditarily aspherical* if it can be embedded in the Hilbert cube Q in such a way that for each $\epsilon > 0$ there exists an ϵ-covering $\mathcal{U}$ of Z by open subsets of Q, where the union of any subcollection of elements of $\mathcal{U}$ is aspherical.

To show, in Section 4, that the boundaries of 7-systolic groups are strongly hereditarily aspherical, we will use the following result.

Proposition 2.9 *[Da, Proposition 1] Suppose* $\{L_i, \mu_i\}$ *is an inverse sequence of finite polyhedra and PL maps, and suppose each* L_i *is endowed with a fixed triangulation* T_i *such that*

1. μ_i^{-1}(each subcomplex of L_i) *is aspherical, and*
2. *there exists a sequence* $(a_k)_{k=1}^{\infty}$ *of positive numbers, such that* $\lim_{k\to\infty} a_k = 0$*, and* $\operatorname{diam}(\mu_{i-k} \circ ... \circ \mu_{i-1}(\sigma)) < a_k$*, for every simplex* $\sigma \in T_i$*.*

Then the inverse limit $Z = \text{inv lim}\{L_i, \mu_i\}$ *is strongly hereditarily aspherical.*

It should be noticed that, by Daverman and Dranishnikov [DaDr, Theorem 2.10], every strongly hereditarily aspherical compactum can be expressed as an inverse limit like the one above.

3 7-systolic complexes

In this section we study some properties of 7-systolic complexes. In particular we define and examine other (then in the general systolic case) contractions on spheres. This is crucial for Section 4. Then we study the properties of some 7-systolic chamber complexes. Those results are important for Section 5.

Let X be a 7-systolic complex of dimension $n < \infty$. Let Q be its convex subcomplex (see [JS1, Sections 3 and 7] and compare Section 2.2 above). For a natural number k, we denote by S_k the combinatorial sphere $S_k(Q, X)$ (compare Section 2.2 above) and we denote by B_k the closed ball $B_k(Q, X)$.

Define a map $\pi_{Q,k}\colon S_k^{(0)} \to (S'_{k-1})^{(0)}$ by putting $\pi_{Q,k}(w) = b_\tau$, for every vertex w of S_k, where $\tau = X_w \cap B_{k-1}$ is the projection of w on S_{k-1} and $b_\tau \in (S'_{k-1})^{(0)}$ is the barycenter of τ.

Lemma 3.1 *Let* v_1 *and* v_2 *be two vertices in* S_k *belonging to the same simplex. Then* $\pi_{Q,k}(v_1)$ *and* $\pi_{Q,k}(v_2)$ *belong to the same simplex of* S'_{k-1}.

Proof: Let $\tau = X_{<v_1,v_2>} \cap B_{k-1}$ and $\tau_i = X_{v_i} \cap B_{k-1}$, for $i = 1, 2$. Then $\tau \subset \tau_1 \cap \tau_2$. It is enough to show that $\tau_1 \subset \tau_2$ or $\tau_2 \subset \tau_1$.

We will show this arguing by contradiction. Suppose it is not true, i.e. there exist vertices w_i such that $w_i \in \tau_i \setminus \tau_j$ for $\{i, j\} = \{1, 2\}$. Let for $i = 1, 2$, t_i be vertex belonging to $(X_{w_i} \cap B_{k-2}) \cap (X_\tau) \cap B_{k-2})$.

Let us examine the closed path $(v_1, w_1, t_1, t_2, w_2, v_2, v_1)$ in $X^{(1)}$.

There are no diagonals of the form $< v_i, t_j >$ since the distance between v_i and t_j is at least 2, $i, j = 1, 2$.

There are no diagonals of the form $< v_i, w_j >$, $i \neq j$. Since if, e.g. $< v_1, w_2 >$ is an edge in X then $w_2 \in X_{v_1} \cap B_{k-1} = \tau_1$.

There is no diagonal $< w_1, w_2 >$. If it exists, then the path $(v_1, w_1, w_2, v_2, v_1)$ is a closed simple path without diagonals of length 4 which contradicts 7-largeness of X.

Hence the path $(v_1, w_1, t_1, t_2, w_2, v_2, v_1)$ is a closed path of length at most six without diagonals. This contradicts 7-largeness of X. $\diamond$

Using the above lemma we can extend $\pi_{Q,k}$ simplicially.

Definition 3.2 *Define, for a natural number* k, *a contiunous map between combinatorial spheres*

$$\pi_{Q,k}\colon S_k(Q, X) \to (S_{k-1}(Q, X))',$$

given by a simplicial extension of the map

$$\pi_{Q,k}\colon S_k(Q, X)^{(0)} \to (S_{k-1}(Q, X)')^{(0)}.$$

Lemma 3.3 *There exists a constant $C < 1$, depending only on $n = \dim(X)$ such that for every $k, l \in \{1, 2, 3, ...\}$ with $l < k$ and for every two points $x, y \in S_k$ one has $\mathrm{d}_{S_{k-l-1}}(\pi_{Q,k-l} \circ ... \circ \pi_{Q,k-1} \circ \pi_{Q,k}(x), \pi_{Q,k-l} \circ ... \circ \pi_{Q,k-1} \circ \pi_{Q,k}(y)) \leq C^l \mathrm{d}_{S_k}(x, y)$.*

Proof: Let D be the distance from a vertex to an opposite codimension one face in the regular n-simplex. Let E be the diameter of a maximal simplex in the barycentric subdivision of the regular n-simplex. Then for $C = \frac{E}{D} < 1$ the lemma holds. $\diamond$

Lemma 3.4 *For every subcomplex L of S_{k-1} the subcomplex $\pi_{Q,k}^{-1}(L)$ of S_k is aspherical.*

Proof: We will show that $\pi_{Q,k}^{-1}(L)$ is a full subcomplex of S_k. Hence, by Proposition 2.1 , as a full subcomplex of 6-large complex it is 6-large and thus aspherical, by Theorem 2.5.

Let vertices $v_1, v_2, ..., v_l \in (\pi_{Q,k}^{-1}(L))^{(0)}$ span a simplex in S_k. Then, by Lemma 3.1, $\pi_{Q,k}(v_1), ..., \pi_{Q,k}(v_l)$ are vertices of a simplex of S'_{k-1} and they correspond to a chain of simplices $\tau_1, ..., \tau_l$ of S_{k-1}. One of them, say τ_1 is the least dimensional simplex among them and hence it is contained in all the simplices $\tau_2, ..., \tau_l$. This means that $\pi_{Q,k}^{-1}(\tau_1) \subset \pi_{Q,k}^{-1}(L)$ contains all points $v_1, ..., v_l$ and hence simplex spanned by them. $\diamond$

In the rest of this section we study some 7-large chamber complexes.

For a 7-large chamber complex X we define, for a vertex v of X, a condition $R(v, X)$ that will be crucial for Section 5 (compare the condition $R(v, X)$ defined in [O, Section 4]):

$$R(v, X) \text{ iff } (\forall\, \sigma \in X_v \ (X_v \setminus \overset{\circ}{B}_2(\sigma, X_v) \text{ and } X_v \setminus \overset{\circ}{B}_3(\sigma, X_v) \text{ are connected}))$$

The next lemma is an analogue of [O, Lemma 4.7], for 7-large complexes.

Lemma 3.5 *Let X be a 7-large chamber complex such that the link X_κ is connected, for every simplex κ of X of codimension greater than one, and $X_\sigma \setminus \overset{\circ}{B}_i(\rho, X_\sigma)$, $i = 2, 3$ is connected for every codimension two simplex σ of X and every simplex ρ of its link X_σ. Then for every vertex v of X condition $R(v, X)$ holds.*

Proof: We will proceed by induction on $n = \dim(X)$.

For $n = 2$ the assertion is clear since codimension two simplexes are just vertices.

Assume we proved the lemma for $n \leq k$. Let $\dim(X) = k + 1$. Take a vertex v of X and consider its link X_v. It has dimension k. Moreover for every codimension l simplex σ of X_v the simplex $\sigma * v$ is of codimension l in X and $X_{\sigma * v} = (X_v)_\sigma$. Thus X_v satisfies hypotheses of the lemma. Hence by the induction assumptions, for every vertex w of X_v condition $R(w, X_v)$ holds.

X_v is 7-large as a full subcomplex of X (Proposition 2.1). Thus the universal cover $\widetilde{X_v}$ of X_v is 7-systolic. Let $p : \widetilde{X_v} \to X_v$ be a covering map.

Take a simplex ω of X_v. Let κ be a simplex of $\widetilde{X_v}$ such that $p(\kappa) = \omega$. Since $\partial\kappa = S_0(\kappa, \widetilde{X_v})$ is connected we have, by [O, Lemma 4.5] (or by Corollary 5.4 below), that $S_1(\kappa, \widetilde{X_v})$, $S_2(\kappa, \widetilde{X_v})$ and $S_3(\kappa, \widetilde{X_v})$ are connected. By Lemma 2.6, $p' = p|_{B_2(\kappa, \widetilde{X_v})} : B_2(\kappa, \widetilde{X_v}) \to B_2(\omega, X_v)$ is an isomorphism. Observe that $p(B_3(\kappa, \widetilde{X_v})) \subset B_3(\omega, X_v)$. We want to show that $p(B_3(\kappa, \widetilde{X_v})) = B_3(\omega, X_v)$. Let z be a vertex in $B_3(\omega, X_v) \setminus B_2(\omega, X_v)$ and let $u \in B_2(\omega, X_v)$ be a vertex connected by an edge with z. Then, by Lemma 2.6, $p'' = p|_{B_1((p')^{-1}(u), \widetilde{X_v})} : B_1((p')^{-1}(u), \widetilde{X_v}) \to B_1(u, X_v)$ is an isomorphism and $(p'')^{-1}(z) \in B_3(\kappa, \widetilde{X_v})$. Hence $z \in p(B_3(\kappa, \widetilde{X_v}))$ and $p(B_3(\kappa, \widetilde{X_v})) = B_3(\omega, X_v)$. Now we claim that $S_3(\omega, X_v) = p(S_3(\kappa, \widetilde{X_v}))$ and hence is connected. Observe that $S_3(\omega, X_v) \subset p(S_3(\kappa, \widetilde{X_v}))$. Suppose $S_3(\omega, X_v) \neq p(S_3(\kappa, \widetilde{X_v}))$. Let $w_1 \in S_3(\kappa, \widetilde{X_v})$ be a vertex such that $p(w_1) \in B_2(\omega, X_v)$. The vertex $w_2 = (p')^{-1}(p(w_1))$ belongs to $B_2(\kappa, \widetilde{X_v})$ and $p(w_1) = p(w_2)$. But then $d_{\widetilde{X_v}^{(1)}}(w_1, w_2) < 7$ and we can find homotopically non-trivial closed path of length less than 7 in X_v. This contradicts 7-largeness of X_v, by [JS1, Corollary 1.5]. Thus we have shown that $S_3(\omega, X_v) = p(S_3(\kappa, \widetilde{X_v}))$ is connected.

Take two vertices t and s of $X_v \setminus \overset{\circ}{B}_2(\omega, X_v)$ (or of $X_v \setminus \overset{\circ}{B}_3(\omega, X_v)$). Since, by assumptions, X_v is connected there exists a path in $(X_v)^{(1)}$ joining them. If this path misses $\overset{\circ}{B}_2(\omega, X_v)$ (respectively $\overset{\circ}{B}_3(\omega, X_v)$) it joins these vertices in $X_v \setminus \overset{\circ}{B}_2(\omega, X_v)$ (respectively in $X_v \setminus \overset{\circ}{B}_3(\omega, X_v)$). If not we can replace it, by connectedness of $S_2(\omega, X_v)$ (respectively $S_3(\omega, X_v)$), by a path intersecting $S_2(\omega, X_v)$ (respectively $S_3(\omega, X_v)$) and also lying in $X_v \setminus \overset{\circ}{B}_2(\omega, X_v)$ (respectively in $X_v \setminus \overset{\circ}{B}_3(\omega, X_v)$). Hence one gets the conclusion. $\diamond$

Corollary 3.6 *Let X be a normal 7-systolic pseudomanifold. Then condition $R(v, X)$ holds for every vertex v of X.*

Proof: One dimensional link in a normal pseudomanifold is a circle. Hence it satisfies assumptions of the preceding corollary. $\diamond$

4 Gromov boundary

The aim of this section is to prove that the ideal boundary of a 7-systolic group (such groups are word hyperbolic by Theorem 2.2) is a strongly hereditarily aspherical compactum. To prove this we first show that such a boundary can be described as an inverse limit of combinatorial spheres in the complex on which the group acts geometrically.

Throughout this section X denotes a locally finite 7-systolic complex of dimension $n < \infty$. We fix a vertex v of X. For a natural number k, we denote by S_k

the combinatorial sphere $S_k(v,X)$ and we denote by B_k the closed ball $B_k(v,X)$. We denote by π_k the projection $\pi_{\{v\},k}\colon S_k \to S_{k-1}$ (see Section 3).

Lemma 4.1 *$\delta X = \operatorname{inv\,lim}\{S_k, \pi_k\}$ is homeomorphic to ∂X–the Gromov boundary of X.*

Proof: We use the set of equivalence classes of geodesic rays in $X^{(1)}$ propagating from a given vertex v, as a definition of the Gromov boundary of X—for details see e.g. Bridson and Haefliger [BrH, Chapter III.3].

Compactness of both δX and ∂X follows from the fact that the balls in $X^{(0)}$ are finite.

First, we construct a bijection $F\colon \delta X \to \partial X$. Let $x = (v, x_1, x_2, ...) \in \delta X$. Note that $x_k \in S_k$ for $k = 1, 2, ...$ For arbitrary k, choose a maximal simplex σ_k of S_k containing x_k. If we take a vertex u of σ_k then $X_{\sigma_k} \cap B_{k-1} \subset X_u \cap B_{k-1}$ and hence there exists a vertex w of σ_{k-1} connected by an edge with u. Hence for any k we can construct a sequence $(v = v_0^k, v_1^k, v_2^k, ..., v_k^k)$ of vertices of X such that $v_i^k \in \sigma_i$ and v_k^{i-1} is connected by an edge with v_k^i for $i = 1, 2, ..., k$. Since a path $c_k = (v_0^k, ..., v_k^k)$ in $X^{(1)}$ has length k and joins v and v_k^k lying at a distance k it is a geodesic segment starting at v. Since the balls in $X^{(0)}$ are finite, we can, by the diagonal argument, extract from $(c_k)_{k=1}^{\infty}$ a geodesic ray $c = (v, v_1, v_2, ...)$, such that v_k is a vertex of σ_k, for every $k = 1, 2, 3, ...$ The equivalence class of c within ∂X is by definition $F(x)$. Observe that it is independent of choosing c as above, since all of them lie at distance at most one from the sequence x.

We show now that F is injective. Let $x = (v, x_1, x_2, ...)$ and $y = (v, y_1, y_2, ...)$ be two elements of δX with $F(x) = F(y)$. Let the geodesic rays $c = (v, v_1, v_2, ...)$ and $d = (v, w_1, w_2, ...)$ in $X^{(1)}$ representing, respectively, $F(x)$ and $F(y)$ be constructed as above. Then there exists a constant $D > 0$ such that for every $k = 1, 2, 3, ...$ we have $d_{S_k}(v_k, w_k) \leq D$. Fix k. It is enough to show that $d_{S_k}(x_k, y_k) \leq \epsilon$ for every $\epsilon > 0$. Choose $\epsilon > 0$. Take $l \in \mathbf{N}$ such that $l \geq \log_C \frac{\epsilon}{D+2}$, where $C < 1$ is the constant of Lemma 3.3. By construction $d_{S_{k+l}}(x_{k+l}, y_{k+l}) \leq D + 2$ and thus by Lemma 3.3

$$d_{S_k}(x_k, y_k) =$$

$$= d_{S_k}(\pi_{k+1} \circ ... \circ \pi_{k+l-1} \circ \pi_{k+l}(x_{k+l}), \pi_{k+1} \circ ... \circ \pi_{k+l-1} \circ \pi_{k+l}(y_{k+l})) \leq$$

$$\leq C^l d_{S_{k+l}}(x_{k+l}, y_{k+l}) \leq \epsilon.$$

Now we show F is onto. Take a geodesic ray $c = (v, v_1, v_2, ...)$, $v_i \in X^{(0)}$. Observe that $v_k \in S_k$ for $k = 2, 3, 4, ...$ Consider a sequence $(\pi_2 \circ \pi_3 \circ ... \circ \pi_k(v_k))_{k=2}^{\infty}$ of points in S_1. By compactness of spheres there is a subsequence $(v_{a^1(1)}, v_{a^1(2)}, ...)$ of the sequence $(v_2, v_3, ...)$ such that $(\pi_2 \circ \pi_3 \circ ... \circ \pi_{a^1(k)}(v_{a^1(k)})_{k=2}^{\infty}$ converges. Let $x_1 \in S_1$ be the limit of this subsequence. Now given a subsequence (of $(v, v_1, v_2, ...)$) $(v_{a^l(1)}, v_{a^l(2)}, ...)$, $l > 1$ we find a subsequence $(v_{a^{l+1}(1)}, v_{a^{l+1}(2)}, ...)$, $a^{l+1}(i) > l$ such that $(\pi_{l+1} \circ \pi_{l+2} \circ ... \circ \pi_{a^{l+1}(k)}(v_{a^{l+1}(k)})_{k=l+1}^{\infty}$ tends to $x_l \in S_l$. By construction $\pi_k(x_k) = x_{k-1}$, for $k > 1$ and $\pi_1(x_1) = v$. Hence $x = (v, x_1, x_2, ...) \in \delta X$. Moreover since v_k and v_{k+1} belong to a common simplex for every $k = 2, 3, 4...$ we get, by definition of π_k, that $d(v_k, \pi_{k+1}(v_{k+1})) \leq C$ where C is the constant from Lemma 3.3. Then for every natural number l we have $d(v_k, \pi_{k+1} \circ \pi_{k+2} \circ ... \circ \pi_{k+l}(v_{k+l})) \leq$

$\sum_{i=1}^{\infty} C^i < \infty$. Thus $d(v_k, x_k) < \sum_{i=1}^{\infty} C^i < \infty$, that implies c represents the equivalence class of $F(x)$.

Finally we argue F is continuous and hence as a continuous bijection defined on a compact space it is a homeomorphism.

Given $x = (v, x_1, x_2, ...) \in \delta X$ and a sequence $(x^i)_{i=1}^{\infty} \subset \delta X$, $x^i = (v, x_1^i, ...)$ converging to x, fix geodesic rays $c = (v, v_1, v_2, ...)$ and $c_i = (v, v_1^i, v_2^i, ...)$, $i = 1, 2, 3, ...$ representing, respectively, $F(x)$ and $F(x^i)$, $i = 1, 2, 3, ...$, and constructed as when we defined F. To prove F is continuous at x it is enough to show that for every natural number N there exists $M > 0$ such that for every natural number $i > M$ we have $d_{S_N}(v_N, v_N^i) < 3$. By definition of the topology of an inverse limit there exists $M > 0$ such that for every natural number $i > M$ one has $d_{S_N}(x_N, x_N^i) < 1$ and hence $d_{S_N}(v_N, v_N^i) \leq d_{S_N}(v_N, x_N) + d_{S_N}(x_N, x_N^i) + d_{S_N}(x_N, v_N^i) < 3$. $\diamond$

We now state and prove the following main theorem.

Theorem 4.2 *The ideal boundary of a 7-systolic group is a strongly hereditarily aspherical compactum.*

Proof: A 7-systolic group G acts, by definition, geometrically on a locally finite 7-systolic complex X of finite dimension. Then the ideal boundary ∂G of G is homeomorphic to ∂X.

We apply Proposition 2.9 to the inverse system $\{L_i, \mu_i\} = \{S_i, \pi_{i+1}\}$. By Lemma 3.4 the condition 1) of Proposition 2.9 is fulfilled, and by Lemma 3.3 we get condition 2) of the proposition. Hence $\partial G = \partial X = \delta X = \text{inv lim}\,\{S_i, \pi_{i+1}\}$ is a strongly hereditarily aspherical compactum. $\diamond$

Remarks. **1)** A simple argument shows that every compact metrizable space can be homeomorphic to the ideal boundary of some hyperbolic space (even more—of some $CAT(-1)$ space). The question of which topological spaces can occur as boundaries of hyperbolic groups (compare Benakli and Kapovich [BeK, Chapter 17]) is more difficult. It is answered somehow only in dimensions (of the boundary) 0 and 1 (cf. Kapovich and Kleiner [KaKl]). For higher dimensions the following homeomorphism types of the boundaries of hyperbolic groups were known: spheres, Pontryagin surfaces Π_p for p being a prime number, two-dimensional universal Menger compactum μ_2^5 (compare [BeK, Chapter 17]), three-dimensional universal Menger compactum μ_3^7 (cf. Dymara and Osajda [DyO]), Pontryagin spheres and three-dimensional trees of manifolds (cf. Przytycki and Świątkowski[PS]).

By Theorem 2.2, 7-systolic groups are hyperbolic and, by [JS1, Corollary 19.3], for each natural number n, there exists a hyperbolic group of cohomological dimension n. Hence, by Theorem 4.2 above, and by Bestvina and Mess [BesM, Corollary 1.4], strongly hereditarily aspherical compacta of all dimensions can occur as boundaries of hyperbolic groups.

Moreover, in [JS1] examples of 7-systolic groups acting on pseudomanifolds of arbitrary large dimension are constructed. Thus, by Corollaries 5.7 and 5.9, those group are, in a sense, indecomposable and their boundaries are connected, locally connected and without local cut points (compare Section 5).

2) Paweł Zawiślak has shown (private communication) that the boundary of a 7-systolic orientable normal pseudomanifold of dimension 3 is the Pontryagin sphere (cf. Jakobsche [Jak]). Such pseudomanifolds are constructed in [JS1].

The Pontryagin sphere is the inverse limit of an inverse system $\{X_i, p_i\}_{i=1}^{\infty}$ defined as follows. Let $X_1 = S^2$ be a triangulated two-sphere. Let $\mathcal{T}$ be a given triangulation of the two torus. Assume X_i, p_j are defined for $i \leq k$ and $j \leq k-1$. Let X_k be a surface and $\mathcal{T}_k$ its triangulation. X_{k+1} is a connected sum of X_k and a set of disjoint tori T_σ–one for every 2-simplex σ of $\mathcal{T}_k$–carrying the triangulation $\mathcal{T}$. Every T_σ is glued to X_k by identifying $\partial\sigma$ and the boundary of some 2-simplex σ' of triangulation of T_σ. Then X_{k+1} carries an induced triangulation and we define a triangulation $\mathcal{T}_{k+1}$ of X_{k+1} as a subdivision of this natural triangulation. The map $p_k \colon X_{k+1} \to X_k$ is defined by the conditions: $p_k(T_\sigma \setminus \sigma') = \text{int}\,\sigma$ and $p_k|_{\partial\sigma} = Id_{\partial\sigma}$ for every 2-simplex σ of $\mathcal{T}_k$.

3) For a polytopal complex Y its *face complex* $\Phi(Y)$ is a simplicial complex defined as follows. The vertex set of $\Phi(Y)$ is the set of cells of Y and the vertices of $\Phi(Y)$ span a simplex if the cells of Y corresponding to those vertices are contained in a common cell of Y. It can be shown (compare [JS3]) that if Y is a simply connected simple (i.e. all links are simplicial complexes) polytopal complex with 7-large links then $\Phi(Y)$ is 7-systolic. Thus the ideal boundary of such a complex Y is strongly hereditarily aspherical. **Question.**

Is the ideal boundary of a hyperbolic systolic group strongly hereditarily aspherical ?

5 Splittings

The aim of this section is to study further properties of boundaries of 7-systolic complexes in some special cases. As a consequence we get results concerning splittings of groups acting on such complexes.

Throughout this section X denotes a locally finite 7-systolic complex of dimension $n < \infty$. We fix a vertex v of X. For a natural number k, we denote by S_k the combinatorial sphere $S_k(v, X)$ and we denote by B_k the closed ball $B_k(v, X)$. We denote by π_k the projection $\pi_{\{v\},k} \colon S_k \to S_{k-1}$ (see Section 3).

Lemma 5.1 *Let Y be a 7-large n-dimensional chamber complex, σ one of its simplices and τ an $n-1$-simplex of $S_2(\sigma, Y)$. Then there exists a vertex $v \in Y \setminus B_2(\sigma, Y)$ such that $v * \tau$ is a simplex of Y.*

Proof: By Lemma 2.8, if we consider the universal cover $p : \widetilde{Y} \to Y$ and $\widetilde{\sigma} \in p^{-1}(\sigma)$, $\widetilde{\tau} \in p^{-1}(\tau) \cap S_2(\widetilde{\sigma}, \widetilde{Y})$, then there exists a vertex $\widetilde{v}$ of $\widetilde{Y}$ such that $\widetilde{v} \in \widetilde{Y} \setminus B_2(\widetilde{\sigma}, \widetilde{Y})$ and $\widetilde{v} * \widetilde{\tau}$ is a simplex of $\widetilde{Y}$. Consider $v = p(\widetilde{v})$. Clearly $v * \tau$ is a simplex of Y.

Assume $v \in B_2(\sigma, Y)$. Then there exists a simplex $\widetilde{\sigma_1} \in p^{-1}(\sigma)$ distinct from $\widetilde{\sigma}$ such that $\widetilde{v} \in B_2(\widetilde{\sigma_1}, \widetilde{Y})$. Since $\widetilde{v} \in B_3(\widetilde{\sigma}, \widetilde{Y})$ we can then choose vertices $s \in \widetilde{\sigma}$ and $t \in \widetilde{\sigma_1}$ with $p(s) = p(t)$ and a path of length at most 6 joining s and t. But this contradicts 7-largeness of Y. Thus $v \in Y \setminus B_2(\sigma, Y)$. $\diamond$

Lemma 5.2 *The map $\pi_k\colon S_k \to S_{k-1}$ is onto.*

Proof: Let z be a given point in S_{k-1}. We will show that there exists a point $w \in S_k$ satisfying $\pi_k(w) = z$.

Case 1: z is a barycenter of a simplex τ of S_{k-1}.

If $\dim(\tau) = n-1$ then by Lemma 2.8 there exists a vertex $w \in S_k$ such that $w * \tau$ is a simplex of X and hence $\pi_k(w) = z$.

Now, let $\dim(\tau) = m < n-1$. Since, by Lemma 2.7, S_{k-1} is a chamber complex of dimension $n-1$, there exists an $n-1$-simplex ρ of S_{k-1} containing τ. Then, again by Lemma 2.8, there exists a vertex $w' \in S_k$ spanning a simplex with ρ. Clearly $w' \in S_k \cap S_2(X_\tau, \delta)$, where $\delta = X_\tau \cap S_{k-2}$. Hence X_τ is a 7-large $(n-m-1)$-dimensional chamber complex and, (again by Lemma 2.7) $S_2(X_\tau, \delta)$ is an $(n-m-2)$-dimensional chamber complex (nonempty), we get, by Lemma 5.1, that there exists a vertex $w \in X_\tau \setminus B_2(X_\tau, \delta)$. But then $\pi_k(w) = z$.

Case 2: z belongs to an interior of an m-simplex τ of S'_{k-1}.

Then $\tau =< a_0, a_1, ..., a_m >$ where a_i is a barycenter of an i-simplex τ_i of S_{k-1}. By *Case 1* there exists a vertex $a'_m \in S_k$ such that $\pi_k(a'_m) = a_m$. Then $a'_m \in S_2(X_{\tau_{m-1}} \cap B_{k-2}, X_{\tau_{m-1}})$ and, using Lemma 5.1 for $X_{\tau_{m-1}}$, there exists a vertex $a'_{m-1} \in X_{\tau_{m-1}} \setminus B_2(X_{\tau_{m-1}} \cap B_{k-2}, X_{\tau_{m-1}})$ connected to a'_m by an edge. Note that $\pi_k(a'_{m-1}) = a_{m-1}$ and that $< a'_m, a'_{m-1} >\subset S_2(X_{\tau_{m-2}} \cap B_{k-2}, X_{\tau_{m-2}})$. Assume we found vertices $a'_m, a'_{m-1}, ..., a'_l$, $l > 0$ spanning a simplex in $S_2(X_{\tau_{l-1}} \cap B_{k-2}, X_{\tau_{l-1}})$, such that $\pi_k(a'_i) = a_i$. Then we can find a vertex $a'_{l-1} \in X_{\tau_{l-1}} \setminus B_2(X_{\tau_{l-1}} \cap B_{k-2}, X_{\tau_{l-1}})$ spanning together with $< a'_m, a'_{m-1}, ..., a'_l >$ a simplex in X. Hence wee can find points $a'_m, a'_{m-1}, ..., a'_0 \in S_k$ spanning a simplex in X and satisfying $\pi_k(a'_i) = a_i$. Then if $z = \sum_{i=0}^m \lambda_i a_i$ for $\lambda_i > 0$ such that $\sum_{i=0}^m \lambda_i = 1$, we have $\pi_k(\sum_{i=0}^m \lambda_i a'_i) = z$. $\diamond$

Lemma 5.3 *Let the condition $R(w, X)$ hold for every vertex w of X. Then $\pi_k^{-1}(\tau)$ is connected for every simplex $\tau \subset S_{k-1}$ and for every $k \geq 2$.*

Proof: If τ is a vertex then its preimage by the map $\pi_k\colon S_k \to S_{k-1}$, $\pi_k^{-1}(\tau) = \text{span}\{\text{vertices in } X_\tau \setminus B_2(X_\tau \cap B_{k-2}, X_\tau)\}$ is nonempty, by Lemma 5.2 and it is connected by $R(\tau, X)$.

Then for the general case, by surjectivity of π_k (Lemma 5.2), it is enough to prove the following. Let z be a given vertex of τ and $p \in S_k$ be a point such that $\pi_k(p)$ belongs to a simplex ρ of S'_{k-1} containing z. Then one can connect p with $\pi_k^{-1}(z)$ by a path.

To prove this claim notice that $\rho =< z = w_0, w_1, w_2, ..., w_m >$ for some vertices $w_1, ..., w_m$ of S'_{k-1} such that $\pi_k(p) = \sum_{i=0}^m \lambda_i w_i$ where $\lambda_i > 0$ and $\sum_{i=0}^m \lambda_i = 1$. Then, by Definition 3.2 of π_k, there exist vertices $v'_0, w'_1, w'_2, ..., w'_m$ spanning a simplex of S_k such that $p = \sum_{i=0}^m \lambda_i w'_i$ and $\pi_k(w'_i) = w_i$. $\diamond$

Corollary 5.4 *Let the condition $R(w, X)$ hold X for every vertex w of X. Then for every $k \geq 2$ and for any connected subcomplex K of S_{k-1} its preimage $\pi_k^{-1}(K)$ is connected.*

Theorem 5.5 *Let X be a finitely dimensional locally finite 7-systolic chamber complex such that the condition $R(w, X)$ holds for every vertex w of X. Then the ideal boundary ∂X of X is connected.*

Proof: Observe that $S_1 = X_v$ and thus it is connected by $R(v, X)$. By Corollary 5.4 if S_{k-1} is connected then S_k is connected too. Hence ∂X as an inverse limit of continua is a continuum. $\diamond$

Theorem 5.6 *Let X be a locally finite 7-systolic chamber complex of finite dimension $n \geq 3$. Assume that the link X_κ is connected, for every simplex κ of X of codimension greater then one, and $X_\sigma \setminus \overset{\circ}{B}_i\ (\rho, X_\sigma)$, $i = 2, 3$ is connected for every codimension two simplex σ of X and every simplex ρ of its link X_σ. Then the ideal boundary ∂X of X is connected and has no local cut points.*

Proof: Connectedness of the boundary follows from Lemma 3.5 and Theorem 5.5.

Now we show there are no local cut points in ∂X. If a point $x \in \partial X$ disconnects an open connected set $U \subset \partial X$ then it disconnects every open connected $V \subset U$. Hence it disconnects every connected subset W whose interior contains x. Thus, to prove the Lemma, it is enough to show that for a given point $x \in \partial X$ and its open neighbourhood U there exists a connected set W with $x \in int\, W \subset U$ such that $W \setminus \{x\}$ is connected.

Let us define, for a natural number k, a map $\pi_k^\infty : \partial X \to S_k$ as a projection from the inverse limit ∂X to the element S_k of the inverse system $\{S_i, \pi_i\}$. By the definition of the topology on ∂X one can find a natural number k large enough so that if τ is a simplex of S_k containing $\pi_k^\infty(x)$ then $W = (\pi_k^\infty)^{-1}(B_2(\tau, S_k)) \subset U$. We claim W is as desired.

First observe that $(\pi_k^\infty)^{-1}(\overset{\circ}{B}_2\ (\tau, S_k)) \subset W$ is open and contains x. Moreover $B_2(\tau, S_k)$ is a connected subcomplex of S_k and hence, by Corollary 5.4 the inverse system $\left\{W_l = \pi_l^{-1}(B_2(\tau, S_k), \pi_l|_{W_l}\right\}_{l=k+1}^\infty$ consists of continua and its inverse limit W is a continuum.

Now we show that every two points $y, z \in W \setminus \{x\}$ are connected by a continuum within $W \setminus \{x\}$. Again by the definition of the topology on ∂X we can find m big enough such that there exists a vertex $w \in S_m$ such that $\pi_m^\infty(x) \in \overset{\circ}{B}_1\ (v, S_m)$ and $y, z \notin \overset{\circ}{B}_1\ (v, S_m)$. Since $\pi_m^\infty(W) = \pi_{k+1}^{-1} \circ \pi_{k+2}^{-1} \circ ... \circ \pi_m^{-1}(B_2(\tau, S_k))$ is a connected subcomplex of S_m then, if $S_1(v, S_m) = (S_m)_v$ is connected, we can find a continuum $K \in \pi_m^\infty(W) \setminus \overset{\circ}{B}_1\ (v, S_m)$ connecting y and z. Then $(\pi_m^\infty)^{-1}(K)$ is a continuum (as an inverse limit of continua) in W missing x and containing y and z.

Thus to finish the proof we have to show that $S_1(v, S_m) = (S_m)_v$ is connected. Observe that for every simplex σ of X_v the link of σ in X_v is the link of $\sigma * v$ in X. Hence (compare Lemma 3.5 and its proof) the link X_v is a 7-large chamber complex such that the condition $R(z, X_v)$ holds for every vertex z, provided X_v has dimension above two. Let $\rho = X_v \cap S_{m-1}$. Then, by Lemma 2.4, we get $S_1(v, S_m) = (S_m)_v = X_v \cap S_m = S_1(\rho, X_v)$. Since $\partial\rho$ is connected and balls of

small radii in 6-large complexes are isomorphic with the ones in its universal covers (cf. Lemma 2.6) we get, by Corollary 5.4 that $S_1(\rho, X_v)$ is connected. $\diamond$

Corollary 5.7 *Let X be a locally finite normal 7-systolic pseudomanifold of finite dimension at least 3. Then its ideal boundary ∂X is connected and has no local cut points.*

Proof: One-dimensional links in normal manifolds are circles. $\diamond$

Theorem 5.8 *Let G be a group acting geometrically by automorphisms on a 7-systolic chamber complex X of dimension $n \geq 3$. Assume that the link X_κ is connected, for every simplex κ of X of codimension greater then one, and $X_\sigma \setminus \overset{\circ}{B}_i(\rho, X_\sigma)$, $i = 2, 3$ is connected for every codimension two simplex σ of X and every simplex ρ of its link X_σ. Then G does not split essentially, as an amalgamated product or as an HNN-extension, over a finite nor two-ended group.*

Proof: This follows from Theorem 5.6.

By Stallings theorem [St], G does not split over a finite group—compare also Gromov [G, remarks after Proposition 3.2.A], Coornaert–Delzant–Papadopoulos [CDP, Exercise 4) in Chapter 2] and Ghys and de la Harpe [GdlH, Proposition 17 in Chapter 7.5].

By Bowditch [Bow, Theorem 6.2], G does not split essentially over a two-ended group. $\diamond$

Corollary 5.9 *A group acting geometrically by automorphisms on a locally finite normal 7-systolic pseudomanifold of dimension at least 3 does not split essentially, as an amalgamated product or as an HNN-extension, over a finite nor two-ended group.*

Remarks. **1)** A systolic group acting on a 7-systolic pseudomanifold of dimension at least 3 can split over a surface group (this remark is due to J. Świątkowski). To see this take two isomorphic closed 3-dimensional 7-large pseudomanifolds with links of vertices being closed surfaces (such spaces exist by Januszkiewicz and Świątkowski [JS1, Corollary 19.3 p. (1) and its proof]). Consider complement of an open residue of a given vertex in each of them. The link of the vertex is a convex subcomplex of the complement and hence the union of both complements along that links is 7-large. Thus the fundamental group of the sum splits over the fundamental group of a link which is a surface.

2) As noticed in [O, Section 5] most of the examples of systolic groups—except automorphism groups of complexes of dimension at most two—constructed in [JS1] as fundamental groups of some extra-tileable simplices of groups satisfy assumtions of Theorems 5.6 and 5.8.

At the moment the only 7-systolic groups of virtual cohomological dimension above two known to us are the groups acting on normal 7-systolic pseudomanifolds, constructed in [JS1].

Question.
Can groups acting geometrically on normal 7-systolic pseudomanifolds of dimension at least 3 split over free non-abelian groups ?

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