Systolic groups with isolated flats

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Abstract: We study possible configurations of flats in a systolic complex (a complex with simplicial nonpositive curvature) and apply the results to prove that a systolic complex has the Isolated Flats Property if and only if it does not contain isometrically embedded triplanes. We also show that systolic groups acting geometrically on systolic complexes with the Isolated Flats Property are relatively hyperbolic with respect to their maximal abelian subgroups of rank at least 2.

Keywords: systolic complex, isolated flats, triplane, relatively hyperbolic, asymptotic cone.

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Contents

1. Introduction

Systolic complexes were introduced by Tadeusz Januszkiewicz and Jacek Świątkowski in [JS1] and independently by Frédéric Haglund in [Ha]. They are connected simply connected simplicial complexes, satisfying certain local combinatorial condition (see Definition

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2.1 for details) which is a simplicial analogue of nonpositive curvature. Systolic complexes have many properties similar to properties of CAT(0)-spaces, however systolicity neither implies, nor is implied by nonpositive curvature of the complex equipped with piecewise Euclidean metric for which simplices are regular Euclidean simplices.

In the present paper we study the special class of systolic complexes, analogous to CAT(0)-spaces with the Isolated Flats Property, studied by G. Christopher Hruska. We prove simplicial analogs of the results obtained by Hruska in [Hr1], [Hr2] and [Hr3] for CAT(0)-spaces.

Recall, that a 2-dimensional flat in a systolic complex X is a subcomplex $F \subset X$ isomorphic to a triangulation of a Euclidean plane by congruent equilateral triangles, such that $F^{(1)} \subset X^{(1)}$ is an isometric embedding. By [JS2] there is no systolic triangulation of $\mathbb{E}^n$ for $n > 2$ and systolic complexes do not admit properly discontinuous action of $\mathbb{Z}^n$ for $n > 2$, thus one does not need to consider higher dimensional flats. The systolic complex X satisfies the Isolated Flats Property (IFP) if the diameter of $\mathcal{N}_c(F) \cap \mathcal{N}_c(F')$ for flats F, F' at infinite Hausdorff distance is bounded by a constant dependent only on c (if flats F and F' are at finite distance, then by Theorem 3.2 the distance is at most 1).

The first result of the paper is the characterization of systolic complexes with IFP similar to the one proved by Daniel Wise for 2-dimensional CAT(0)-complexes (see [Hr1]):

Theorem A (see Theorem 5.7 in the text) *Let X be a cocompact systolic complex. Then X satisfies the Isolated Flats Property if and only if it does not contain isometrically embedded triplanes.*

D. Wise defined the triplane as the geodesic space obtained by gluing three half-planes by isometries along their boundaries. In the systolic case there are three configurations resembling triplane – they are shown in Figure 5.1.

The second result is the theorem stating that a group acting geometrically on a systolic complex with IFP is relatively hyperbolic with respect to their maximal virtually abelian subgroups of rank 2 (as we have mentioned above, such groups do not contain abelian subgroups of higher rank).

Theorem B (see Corollaries 5.4 and 5.14 in the text) *Let X be a systolic complex with the Isolated Flats Property and G a group acting cocompactly and properly discontinuously on X. Then:*

(1) *There is a bijective correspondence between maximal virtually abelian subgroups of rank 2 in G and equivalence classes of flats in X (two flats are equivalent if they are at finite Hausdorff distance).*
(2) *The group G is relatively hyperbolic with respect to a family of its maximal virtually abelian subgroups of rank 2.*

These results are summarized in Theorem 5.17, where we present four equivalent formulations of the Isolated Flats Property.

One of the results obtained while proving Theorem A is interesting itself, as it presents a precise description of all possible configurations of two flats in an arbitrary systolic complex (we do not assume IFP until Section 5). A flat (the triangulation of a plane such that every vertex is adjacent to 6 triangles) can be cut into two half-planes such that boundary vertices at each of them are adjacent to 3 triangles (we call them *half-planes of type 3-3*) or such that boundary vertices at each of the half-planes are adjacent alternately to 2 or 4 triangles (we call them *half-planes of type 2-4*). The intersections of two flats in an arbitrary systolic complex are described by the following theorem:

Theorem C (see Theorem 3.13 in the text) *Let F_1 and F_2 be two flats in a systolic complex X. Choose flats F_1' and F_2', at finite Hausdorff distance (the distance is actually at most 1) from F_1 and F_2, respectively, such that the intersection $F_1' \cap F_2'$ is maximal. Then*

$$F_1' \cap F_2' = H_1 \cap \ldots \cap H_n$$

for some collection of half-planes $H_1, \ldots, H_n \subset F$ of type 3-3 or of type 2-4.

2. Systolic complexes and groups

In this section we recall the definition and main properties of systolic complexes and systolic groups, proved in [JS1] and [JS2].

Let X be a simplicial complex and σ a simplex in X. The *link* of X at σ, denoted X_σ, is the subcomplex of X consisting of all simplices that are disjoint from σ and together with σ span a simplex in X. The residuum $Res(\sigma)$ is the union of all simplices containing σ.

A simplicial complex X is *flag* if every finite set of its vertices pairwise connected by edges spans a simplex in X. A subcomplex $Y \subset X$ is *full* if any simplex $\sigma \subset X$ with all vertices in Y is contained in Y.

A *cycle* in X is a subcomplex γ isomorphic to a triangulation of a circle. The *length* of γ (denoted $|\gamma|$) is the number of its edges. *A diagonal* of a cycle is an edge joining its two nonconsecutive vertices.

Whenever we refer to a metric on a simplicial complex, we actually mean the 1-skeleton of the complex equipped with the combinatorial metric (i.e. the geodesic metric in which all edges have length 1). Thus for a simplicial complex X the symbol 'd_X' denotes the combinatorial metric on $X^{(1)}$. Moreover, referring to a geodesic in a simplicial complex X, we mean a geodesic in $X^{(1)}$ having both endpoints in $X^{(0)}$.

Definition 2.1. (see [JS2]) *A simplicial complex X is called:*

- 6-large *if it is flag and every cycle γ in X of length $4 \leq |\gamma| < 6$ has a diagonal;*
- locally 6-large *if link at every (nonempty) simplex in X is 6-large;*
- systolic *if it is locally 6-large, connected and simply connected.*

A group acting simplicially, properly discontinuously and cocompactly on a systolic complex is called a systolic group.

The definition of systolicity emphasizes the local character of simplicial nonpositive curvature; however, we obtain an equivalent definition replacing words 'locally 6-large' with '6-large' (see the fact below).

Fact 2.2. ([JS1], Proposition 1.4) *Every systolic complex is 6-large. In particular, a cycle of length smaller than 6 in a systolic complex bounds a triangulated disc with no internal vertices.*

Theorem 2.3. *Let X be a finite-dimensional systolic complex. Then:*

(1) ([JS1], Theorem 4.1) *X is contractible.*

(2) ([JS2], Corollary 1.3) *Every full subcomplex of X is aspherical.*

The important tool used in the present paper will be the notion of minimal surfaces. The existence of minimal surfaces is given by the following proposition (notice that a minimal surface spanning the given closed path need not be unique). For a more detailed introduction to minimal surfaces in systolic complexes we refer the reader to [E].

Proposition 2.4. ([E], Lemma 4.2) *Let X be a systolic complex and γ a closed path in $X^{(1)}$. Then there exists a simplicial map $S : \Delta \to X$ such that Δ is a triangulation of a 2-disc and $S|_{\partial\Delta}$ coincides with γ. Moreover, if we choose S so that Δ has the minimal area, then Δ is a systolic disc. In the latter case S is called* a minimal surface spanning γ.

Systolic complexes are in some sense essentially 2-dimensional (in the sense of the theorem below). However, there are examples of systolic complexes of arbitrarily large virtual cohomological dimensions (see [JS1]).

Theorem 2.5. ([JS2], Theorem 8.2; [E], Theorem 2.5) *Let X be a systolic complex and S a triangulation of a 2-sphere. Then any simplicial map $f : S \to X$ can be extended to a simplicial map $F : B \to X$, where B is a triangulation of a 3-ball, such that $\partial B = S$ and B has no internal vertices.*

2.1. Directed geodesics

The 1-skeletons of systolic complexes equipped with the combinatorial metric are not uniquely geodesic. Moreover, two geodesics with the same endpoints can be far away from each other (even as far as half of their lengths). To avoid this inconvenience Januszkiewicz and Świątkowski introduced in [JS1] a subclass of geodesics, called *allowable geodesics* (see Definition 2.7) with better properties. An allowable geodesic connecting given two vertices is not unique, but any two such geodesics are Hausdorff 1-close. Moreover, allowable geodesics satisfy some variant of the Fellow Traveller Property:

Proposition 2.6. (Fellow Traveller Property) (see [JS1], Proposition 11.2) *Let X be a systolic complex and suppose $(u_i)_{i=0}^{n}$ and $(t_i)_{i=0}^{m}$ are allowable geodesics in X from v to w and from p to q, respectively. Then:*

$$d_X(u_i, t_i) \leq 3 \cdot \max\{d_X(v, p), d_X(w, q)\} + 1, \quad \textit{for every } i \geq 0$$

(we put $u_{n+j} := u_n$ and $t_{m+j} := t_m$ for $j > 0$).

Unfortunately, the class of allowable geodesics is not symmetric, i.e. an allowable geodesic from u to v may be not allowable when we consider it as a geodesic from v to u. Allowable geodesics are determined by a sequence of simplices in X, called the *directed geodesic*, which has the following local definition:

Definition 2.7. ([JS1]) *For vertices $v, w \in X$, a* directed geodesic *from v to w is a sequence $(\sigma_i)_{i=0}^n$ of simplices, where $\sigma_0 = v$, $\sigma_n = w$, satisfying the following properties:*

(1) *any two consecutive simplices σ_i, σ_{i+1} are disjoint and span a simplex of X;*
(2) *for any three consecutive simplices σ_i, σ_{i+1}, σ_{i+2} we have:*

$$Res(\sigma_i) \cap N(\sigma_{i+2}) = \sigma_{i+1}. \tag{2.1}$$

Any polygonal path in $X^{(1)}$ with consecutive vertices $(u_i)_{i=0}^n$ chosen so that $u_i \in \sigma_i$ is called an allowable geodesic *from v to w.*

Here and subsequently we denote by $N(\sigma)$ the union of all simplices intersecting σ.

Proposition 2.8. *Let X be a systolic complex.*

(1) ([JS1], Corollary 9.8) *Any allowable geodesic in X is a geodesic.*
(2) ([JS1], Corollary 9.7) *For any vertices $v, w \in X$ there is exactly one directed geodesic from v to w. In particular, any two allowable geodesics from v to w are at Hausdorff distance at most 1.*

Example 2.9. A triangulation of the Euclidean plane by congruent equilateral triangles is called *the flat systolic plane* and denoted by $\mathbb{E}^2_\triangle$. We divide the family of directed geodesics in $\mathbb{E}^2_\triangle$ into three types:

- a *directed geodesic of type 3-3* is the sequence of vertices of an arbitrary convex geodesic in the 1-skeleton of $\mathbb{E}^2_\triangle$;
- a *directed geodesic of type 2-4* is the sequence of simplices crossed by a line perpendicular to some convex geodesic (elements of the sequence are, alternately, 0-simplices and 1-simplices);
- a *directed geodesic of mixed type* is a sequence $(\sigma_i)_{i=1}^n$, where $(\sigma_i)_{i=1}^k$ is a directed geodesic of type 2-4, $(\sigma_i)_{i=k}^n$ is a directed geodesic of type 3-3 and the 'Euclidean angle' between them is $\frac{5}{6}\pi$ (as shown in Figure 3.1).

Notice that directed geodesics of type 3-3 or of type 2-4 are symmetric (they remain directed geodesics after reversing the order), while directed geodesics of mixed type are not. We use the above terminology to define half-planes of types 3-3 and 2-4:

Definition 2.10.

- An *allowable geodesic of type 3-3* is a geodesic in $\mathbb{E}^2_\triangle$ disconnecting it into two half-planes whose every boundary vertex is adjacent to 3 triangles.

- An *allowable geodesic of type 2-4* is a geodesic in $\mathbb{E}^2_\triangle$ disconnecting it into two half-planes whose boundary vertices are adjacent alternately to 2 or 4 triangles.

Systolic half-planes obtained in such a way are called *half-planes of type 3-3* and *half-planes of type 2-4*, respectively.

3. Flats in systolic complexes

By *a flat in a systolic complex* X we mean a simplicial map $F : \mathbb{E}^2_\triangle \to X$ which when restricted to the 1-skeleton (with the combinatorial metric) is an isometric embedding into $X^{(1)}$. We will not distinguish between the flat and its image and sometimes refer to a flat as a subcomplex of X.

The detailed study of flats in systolic complexes is presented in [E]. The main results are the following:

Theorem 3.1. ([E], Theorem 5.2) *Let X be a systolic complex and a simplicial map $F : \mathbb{E}^2_\triangle \to X$ be a locally isometric immersion (i.e. the restriction of F to the 1-skeleton of $N(v)$ is an isometric embedding for any vertex $v \in \mathbb{E}^2_\triangle$) such that diam$(\operatorname{Im} F) \geq 3$. Then F is a flat.*

We say that two flats F and F' in a systolic complex are *equivalent* if they are at finite Hausdorff distance. As the following theorem shows, they are actually at distance at most 1 and there is a canonical isometry $\varphi : F \to F'$ such that $d(v, \varphi(v)) \leq 1$ for any vertex v.

Theorem 3.2. ([E], Theorem 5.4) *Let F be a flat in a systolic complex X. Denote by $Th(F) \subset X$ the full subcomplex spanned by all flats at finite Hausdorff distance from F (*the thickening of F*). Then:*

(1) *There is a unique simplicial retraction $r : Th(F) \to F$. Moreover, r restricted to any flat $F' \subset Th(F)$, is an isometry.*
(2) *The counterimage $r^{-1}(v)$ is a simplex of X, for any vertex $v \in F$.*
(3) *If $v, w \in F$ are connected by an edge, then simplices $r^{-1}(v)$ and $r^{-1}(w)$ span a simplex of X.*

In [E] we study connections between flats in a systolic complex X and noncyclic free abelian subgroups in a group G acting cocompactly and properly discontinuously on X, obtaining the simplicial flat torus theorem:

Theorem 3.3. ([E], Theorem 6.1) *Let G be a noncyclic free abelian group acting properly discontinuously by simplicial automorphisms on a uniformly locally finite systolic complex X. Then:*

(1) *G is isomorphic to $\mathbb{Z}^2$.*
(2) *There is a G-invariant flat $F \subset X$, unique up to flat equivalence.*

Theorem 3.4. ([E], Corollary 6.2) *Let a group G act simplicially, cocompactly and properly discontinuously on a systolic complex X.*

(1) *If $H < G$ is a virtually abelian subgroup of rank 2, then there is a flat F, unique up to flat equivalence, such that $Th(F)$ is H-invariant.*
(2) *If $H < G$ is a maximal virtually abelian rank 2 subgroup, then there is a flat F, unique up to flat equivalence, such that $\mathrm{Stab}_G(Th(F)) = H$.*

In Section 6 we will need the following result, which is a special case of main theorems from Section 4 in [E].

Theorem 3.5. (see [E], Theorem 4.13) *Let X be a systolic complex and $\mathcal{N}_r(v) \subset \mathbb{E}^2_\triangle$ a ball about a vertex in $\mathbb{E}^2_\triangle$. If a simplicial map $S : \mathcal{N}_r(v) \to X$ is a minimal surface, then S is an isometric embedding of 1-skeleton of $\mathcal{N}_r(v)$.*

Dealing with minimal surfaces we also need the following consequence of the combinatorial Gauss-Bonnet Theorem:

Lemma 3.6. (Gauss-Bonnet) *If Δ is any triangulation of a 2-disc, then*

$$\sum_{v \in \Delta^{(0)}} \mathrm{def}(v) = 6,$$

where the defect of a vertex is defined as follows:

$$\mathrm{def}(v) = \begin{cases} 6 - \#\{\textit{triangles in } \Delta \textit{ containing } v\}, & \textit{if } v \notin \partial\Delta \\ 3 - \#\{\textit{triangles in } \Delta \textit{ containing } v\}, & \textit{if } v \in \partial\Delta \end{cases}$$

In particular, if Δ is systolic (e.g. it is the domain of a minimal surface), then the sum of defects at boundary vertices is at least 6, with the equality if and only if Δ has no internal vertices of negative defect.

Remark 3.7. (see [E], Remark 3.1) *If Δ is a triangulation of a disc and $g \subset \partial\Delta$ is a geodesic, then the sum of defects along g (i.e. the sum of defects at vertices of g different from the endpoints) is at most 1.*

3.1. Allowable geodesics in flats

Flats in systolic complexes are (by definition) geodesic subcomplexes, but they do not need to be convex (nor even quasi-convex), e.g. the triplane in Figure 5.1(c) is the convex hull of any of the three flats contained in it. However, if we restrict our attention to allowable geodesics (see Definition 2.7), the behaviour of flats is much nicer.

Let $F \subset X$ be a flat in a systolic complex X. Since F is a systolic complex itself, we may consider allowable geodesics in F – to distinguish them from allowable geodesics in X, we will call them F-allowable and X-allowable, respectively. Obviously F-allowable geodesic is a geodesic in X, but it does not need to be X-allowable.

The aim of this section is the proof of the fact (crucial for proofs of Theorems 3.11 and 5.7) that any F-allowable geodesic is 1-close to any X-allowable geodesic with the same endpoints. We prove even more, that there is a flat F' equivalent to F, such that F'-allowability implies X-allowability. Unfortunately, the statement cannot be strengthened, i.e. we cannot say 'is X-allowable geodesic' instead of 'is 1-close to X-allowable geodesic'. For that reason we use equivalent flats in the statement of the following proposition.

Proposition 3.8. *Let $F \subset X$ be a flat in a systolic complex X and $(\sigma_i)_{i=0}^n$ the directed geodesic in X joining vertices $v, w \in F$. Then:*

(1) *There is a flat $F' \subset X$ equivalent to F such that $v, w \in F'$ and the directed geodesic in F' joining v with w, denoted by $(\tau_i')_{i=0}^n$, satisfies*

$$\tau_i' = F' \cap \sigma_i, \text{ for } i = 0, 1, \ldots, n$$

(2) *An F-allowable geodesic is 1-close to any X-allowable geodesic having the same endpoints.*

Proof: We formulate the proposition in a more precise, but more technical way in the following lemma. The proposition follows immediately from the lemma:

Lemma 3.9. *Let $F : \mathbb{E}^2_\triangle \to X$ be a flat in a systolic complex X. Denote by $(\tau_i)_{i=0}^n$ the directed geodesic in $\mathbb{E}^2_\triangle$ joining vertices v with w and by $(\sigma_i)_{i=0}^n$ the directed geodesic in X joining $F(v)$ with $F(w)$. For any i such that $\tau_i \subset \mathbb{E}^2_\triangle$ is a 0-simplex, choose* **arbitrary** *vertex $u_i \in \sigma_i \subset X$. Then there is a flat $F' : \mathbb{E}^2_\triangle \to X$ (equivalent to F) such that:*

(1) *$F'(\tau_i) = u_i$, for i such that τ_i is a 0-simplex in $\mathbb{E}^2_\triangle$,*

(2) *$F'(x) = F(x)$ for vertices $x \in \mathbb{E}^2_\triangle$ not considered in (1),*

(3) *$F'(\tau_i) = \operatorname{Im} F' \cap \sigma_i$, for $i = 0, 1, \ldots, n$.*

Proof: The directed geodesic $(\tau_i)_{n=0}^n$ in $\mathbb{E}^2_\triangle$ has the form shown in Figure 3.1. In particular, $\tau_0, \tau_2, \ldots, \tau_{2k}$ and $\tau_{2k+1}, \tau_{2k+2}, \ldots, \tau_n$ are 0-simplices and $\tau_1, \tau_3, \ldots, \tau_{2k-1}$ are 1-simplices (possibly $k = 0$ or $k = \frac{1}{2}n$).

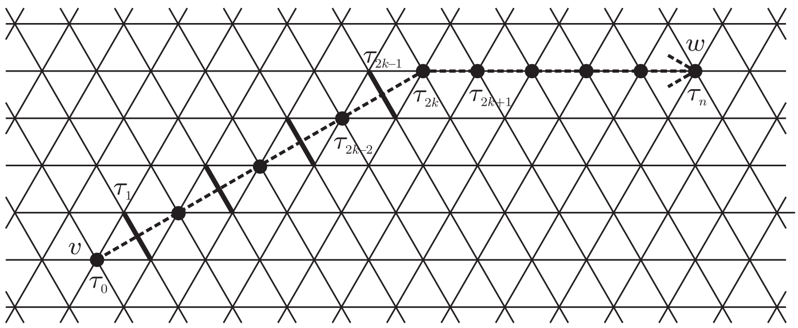

Figure 3.1.

Below we define a sequence of simplices of X and check that it is a directed geodesic joining v with w. By the uniqueness of directed geodesics (Proposition 2.8) we will obtain $\bar{\sigma}_i = \sigma_i$.

If τ_i is a 0-simplex, then we denote by ρ_i the simplex spanned by the vertices $F'(\tau_i)$ for all flats F', such that $F'(x) = F(x)$ for all $x \neq \tau_i$ (it is the simplex $r^{-1}(F(\tau_i))$ presented in Theorem 3.1). Obviously $F(\tau_i) \in \rho_i$. Let us define the sequence $\bar{\sigma}_i$:

$$\begin{array}{rcll} \bar{\sigma}_0 & = & F(v) & \\ \bar{\sigma}_{2m-1} & = & Res(\bar{\sigma}_{2m-2}) \cap N(\rho_{2m}), & \text{for } m = 1, \ldots, k \\ \bar{\sigma}_{2m} & = & Res(\bar{\sigma}_{2m-1}) \cap \rho_{2m}, & \text{for } m = 1, \ldots, k \\ \bar{\sigma}_j & = & \rho_j, & \text{for } j = 2k+1, \ldots, n-1 \\ \bar{\sigma}_n & = & F(w) & \end{array} \tag{3.1}$$

By the definition of ρ_i and Theorem 3.2 we can find a flat F' satisfying (1) and (2) for any vertices $u_i \in \bar{\sigma}_i$. Then by Theorem 3.2 we see that

$$F'(\tau_{2m-1}) \subset \bar{\sigma}_{2m-1}, \text{ for } m = 1, \ldots, k \tag{3.2}$$

so F' satisfies (3) for the sequence $(\bar{\sigma}_i)$.

To prove that $(\bar{\sigma}_i)_{i=0}^n$ is the directed geodesic, we need to check that consecutive elements of the sequence are disjoint simplices, their union span a simplex and that the condition (2.1) holds.

By Theorem 3.2 the simplices ρ_i are pairwise disjoint, ρ_i and ρ_{i+1} span a simplex for $i = 2k, \ldots, n-1$ and $\text{dist}(\rho_i, \rho_{i+2}) = 2$, for any i. Thus $\bar{\sigma}_i$ is a sequence of simplices (to see that $\bar{\sigma}_{2m-1}$ is a simplex we use Fact 2.2 and the fact that $\text{dist}(\rho_i, \rho_{i+2}) = 2$). Simplices $\bar{\sigma}_i$ and $\bar{\sigma}_{i+1}$ are disjoint and span a simplex of X by (3.1) (if $i = 0, \ldots, 2k-1$) or by the fact that ρ_i and ρ_{i+1} span a simplex (if $i = 2k, \ldots, n-1$). Thus

$$\bar{\sigma}_i \subset Res(\bar{\sigma}_{i-1}) \cap N(\bar{\sigma}_{i+1}), \text{ for } i = 1, \ldots, n-1.$$

To prove the opposite inclusion, observe that the inclusion

$$\bar{\sigma}_{2m-1} \supset Res(\bar{\sigma}_{2m-2}) \cap N(\bar{\sigma}_{2m}), \text{ for } m = 1, \ldots, k$$

follows directly from (3.1). To prove the inclusion

$$\bar{\sigma}_j = \rho_j \supset Res(\bar{\sigma}_{j-1}) \cap N(\bar{\sigma}_{j+1}), \text{ for } j = 2k+1, \ldots, n-1$$

it suffices to notice that if a vertex u is connected by edges with two opposite vertices of the isometrically embedded hexagon $F'(N(\tau_j))$ then by Fact 2.2 (applied to two pentagons with a vertex u) it is connected to any vertex of $F'(N(\tau_j))$.

Thus we only need to prove that

$$\bar{\sigma}_{2m} \supset Res(\bar{\sigma}_{2m-1}) \cap N(\bar{\sigma}_{2m+1}), \text{ for } m = 1, \ldots, k,$$

i.e. that any vertex $u \in Res(\bar{\sigma}_{2m-1}) \cap N(\bar{\sigma}_{2m+1})$ is contained in ρ_{2m} (this implies $u \in \bar{\sigma}_{2m}$, as $\bar{\sigma}_{2m} = Res(\bar{\sigma}_{2m-1}) \cap \rho_{2m}$).

Choose such a vertex u and suppose $m < k$. By (3.2) there exist edges ue^0_{2m-1}, ue^1_{2m-1} and edges ye^0_{2m+1}, ye^1_{2m+1}, uy for some vertex $y \in \bar{\sigma}_{2m+1}$ (where e^0_i and e^1_i are the endpoints of the edge $F'(\tau_i)$). Denote the vertices in $N(\tau_{2m-1}) \cap N(\tau_{2m+1}) \subset \mathbb{E}^2_\triangle$, different from τ_{2m}, by a^0_{2m} and a^1_{2m}. By the systolicity of X the pentagons $e^i_{2m-1}uye^i_{2m+1}a^i_{2m}$, for $i = 0, 1$, has two diagonals, however e^i_{2m-1} and e^i_{2m+1} are not connected by an edge (they are at distance 2 in X) and neither are e^i_{2m-1} and y (as by (3.2) and Theorem 3.2 the distance between e^i_{2m-1} and $\bar{\sigma}_{2m+2}$ is 3). Thus the pentagons have the diagonals ua^i_{2m}, $i = 0, 1$. Since u is connected to the opposite vertices $F'(a^0_{2m})$ and $F'(a^1_{2m})$ of the isometrically embedded hexagon $F'(N(\tau_{2m}))$, by Fact 2.2 (applied to two pentagons with vertices u, $F'(a^0_{2m})$, $F'(a^1_{2m})$ and two other vertices of the hexagon $F'(N(\tau_{2m}))$) is connected to all vertices of $F'(N(\tau_{2m}))$, so $u \in \rho_{2m}$.

If $m = k$ we repeat the above argument, proving that u is connected to vertices $F'(e^0_{2k-1})$ and $F'(\tau_{2k+1})$, which are opposite vertices of the hexagon $F'(N(\tau_{2k}))$.

Thus $(\bar{\sigma}_i)$ is the directed geodesic in X joining $F'(v)$ with $F'(w)$, so $\sigma_i = \bar{\sigma}_i$ for $i = 0, \ldots, n$. □

3.2. Quasi-convex geodesics

In this section we prove that the quasi-convexity of a geodesic g in a systolic complex implies quasi-convexity of all geodesics at finite Hausdorff distance from g (Proposition 3.11).

Definition 3.10. *A subset Q of a metric space X is called* δ-quasi-convex *if every geodesic with endpoints in Q is contained in $\mathcal{N}_\delta(Q)$. Subset Q is called* quasi-convex *if it is δ-quasi-convex for some δ.*

Proposition 3.11. *If F is a flat in a systolic complex X and $g \subset F$ a geodesic (finite or infinite) which is convex as a geodesic in F, then g is 1-quasi-convex as a geodesic in X. Moreover, any geodesic g' joining two vertices of g is contained in some flat F' equivalent to F and $g' \subset F'$ is convex as a geodesic in F'.*

Proof: By Theorem 3.2 it suffices to prove the second part of the proposition. Choose vertices $a, b \in g \subset F$ and a geodesic g' in X with endpoints a and b. We prove that there is a flat F' equivalent to F, containing g'. We may assume (not losing generality) that $g \cap g' = \{a, b\}$.

We proceed by induction on the area of Δ, where $S : \Delta \to X$ is a minimal surface spanning the cycle $g^{-1} * g'$. By the Gauss-Bonnet Lemma the sum of defects at boundary vertices of Δ is at least 6, whereas the defects at $S^{-1}(a)$ and $S^{-1}(b)$ are at most 2 and the sums of defects along the geodesics $S^{-1}(g) \subset \Delta$ and $S^{-1}(g') \subset \Delta$ do not exceed 1 (Remark 3.7). Thus the sum of defects along $S^{-1}(g')$ is equal to 1, so there is a vertex $v \in S^{-1}(g) \subset \Delta$ of defect 1.

The vertex v has three neighbours in Δ: $x, y \in S^{-1}(g) \subset \partial\Delta$ and $z \notin S^{-1}(g)$. Thus $S(z) \in X$ is connected by edges with $S(x), S(v), S(y) \in g \subset F$. Sice $S(v)$ is the center of a hexagon in F with opposite vertices $S(x)$ and $S(y)$. Denote the consecutive vertices of the hexagon by $S(x)$, a_0, b_0, $S(y)$, b_1, a_1. By Fact 2.2 the pentagons $S(x)a_ib_iS(y)S(v)$,

$i = 0, 1$ have the diagonals $S(v)a_i$ and $S(v)b_i$, $i = 0, 1$. Thus by replacing $S(v)$ with $S(z)$ we obtain a flat $\bar{F}$ equivalent to F and a convex geodesic $\bar{g} \subset \bar{F}$ such that the area of a minimal surface spanning the cycle $\bar{g}^{-1} * g'$ is smaller than the area of Δ (if $\bar{g}^{-1} * g'$ is not a cycle, but a concatenation of two cycles, we apply the inductive assumption to any of the two cycles). By the inductive assumption g' is contained in a flat F' equivalent to $\bar{F}$, thus equivalent to F and $g' \subset F'$ is convex as a geodesic in F'. □

Proposition 3.12. *If g is a δ-quasi-convex geodesic (finite or infinite) in a systolic complex X and g' is Hausdorff c-close to g, then g' is $(8\delta + 10c + 4)$-quasi-convex.*

Proof: Let $a', b' \in g'$ be arbitrary vertices and $a, b \in g$ be the closest vertices to a' and b', respectively. Denote by $\bar{g} \subset g$ the subgeodesic with endpoints a and b. It suffices to prove that $\mathrm{hdist}_X(\gamma', \bar{g}) \leq 4\delta + 5c + 2$ for any geodesic γ' connecting a' with b'.

First we prove that γ' is close to some geodesic γ with endpoints a and b. Denote by ξ the concatenation $\alpha * \gamma' * \beta^{-1}$ (where α and β are geodesics joining a with a' and b with b', respectively). Then $|\xi| \leq |\bar{g}| + 4c$. We reduce the situation to the case when every geodesic joining a with b intersects ξ only at the endpoints. Let $S : \Delta \to X$ be a minimal surface spanning the closed path $\xi * \gamma^{-1}$, where γ is a geodesic with endpoints a and b such that the area of Δ is minimal. Then the sum of defects at Δ along $S^{-1}(\gamma)$ is nonpositive, so by the Gauss-Bonnet Lemma the sum of defects along $S^{-1}(\xi)$ is at least 2 (in particular $|\xi| > |\gamma|$). Thus either there is a vertex $v \in S^{-1}(\xi)$ of defect 2, or there is a subpath $l \subset S^{-1}(\xi)$ having defects 1 at the endpoints and 0 at other vertices of l. Deleting from Δ triangles adjacent to v or to l and taking the image of the boundary of the new disc we obtain a path ξ_1 which is Hausdorff 1-close to ξ and $|\xi_1| = |\xi| - 1$. Iterating the procedure we finally obtain $|\xi_k| = |\gamma| = |\bar{g}|$ for some $k \leq 4c$, what implies $\xi_k = \gamma$. Thus $\mathrm{hdist}_X(\xi, \gamma) \leq 4c$, so $\mathrm{hdist}_X(\gamma', \gamma) \leq 5c$.

By the assumption $\gamma \subset \mathcal{N}_\delta(g)$, so $\gamma \subset \mathcal{N}_{2\delta}(\bar{g})$. It follows that $\bar{g} \subset \mathcal{N}_{4\delta+2}(\gamma)$ (otherwise there would be a decomposition $\bar{g} = \bar{g}_1 * s * \bar{g}_2$ such that $|s| = 4\delta + 4$ and $\gamma \subset \mathcal{N}_{2\delta}(\bar{g}_1 \cup \bar{g}_2)$, so there would exist an edge $ww' \subset \gamma$, such that $w \in \mathcal{N}_{2\delta}(\bar{g}_1)$ and $w' \in \mathcal{N}_{2\delta}(\bar{g}_2)$, contradicting the geodesity of $\bar{g}$). Therefore $\bar{g}$ and γ' are at Hausdorff distance at most $4\delta + 5c + 2$. □

3.3. Configurations of two flats

Theorem 3.13. *Let F and F' be two flats in a systolic complex X having nonempty maximal intersection (among all pair of flats in equivalence classes $[F]$ and $[F']$). Then*

$$F \cap F' = H_1 \cap \ldots \cap H_n \tag{3.3}$$

for some collection of half-planes $H_1, \ldots, H_n \subset F$, each of type 3-3 or 2-4.

Proof: There are 3 directions of directed geodesics of type 3-3 and 3 directions of directed geodesics of type 2-4 in $\mathbb{E}^2_\triangle$. Consider the relation of being at finite Hausdorff distance on the family of half-planes of type 3-3 and half-planes of type 2-4. There are 12 equivalence classes $\mathcal{H}_1, \ldots, \mathcal{H}_{12}$, each ordered linearly by inclusion. We order the indices such that the 'Euclidean angle' between boundaries of elements of $\mathcal{H}_i$ and $\mathcal{H}_{i+1}$ is $\frac{5\pi}{6}$ (using the cyclic order of indices). Denote by H_i the minimal element of $\mathcal{H}_i$ containing $F \cap F'$ if such exists, or $H_i = \mathbb{E}^2_\triangle$ otherwise, for $i = 1, \ldots, 12$. We prove that $F \cap F' = H_1 \cap \ldots \cap H_{12}$.

The intersection $F \cap F'$ can be equal to the set of 0-simplices of some (finite or infinite) directed geodesic of type 2-4, when it is clearly of the form (3.3). Let us assume this is not the case. Then $H_1 \cap \ldots \cap H_{12}$ is connected.

Step 1: *Let $v, w \in F \cap F'$ be vertices and $(\tau_i)_{i=0}^n$ the directed geodesic from v to w in F. Then every τ_i which is a 0-simplex is contained in $F \cap F'$.*

Let $(\tau_i')_{i=0}^n$ be the directed geodesic from v to w in F' and $(\sigma_i)_{i=0}^n$ the directed geodesic in X. Assume τ_i is a 0-simplex not contained in $F \cap F'$. If τ_i' is also a 0-simplex, then by Lemma 3.9 we replace F and F' with equivalent flats $\bar{F}$ and $\bar{F}'$, obtaining $\bar{\tau}_i = \bar{\tau}_i' \subset \sigma_i$, contradicting the maximality of $F \cap F'$. If τ_i' is a 1-simplex, then by Lemma 3.9 we can replace F with an equivalent flat $\bar{F}$, such that $\bar{\tau}_i \subset \tau_i' \subset \sigma_i$, again obtaining contradiction with the maximality of $F \cap F'$.

Step 2: *For every $H_i \subset F$ the intersection $\partial H_i \cap (F \cap F')$ is connected.*

If H_i is a half-plane of type 3-3, then the statement follows from Step 1. Thus let H_i be a half-plane of type 2-4. Then, by Step 1 and the minimality of H_i, $\partial H_i \cap (F \cap F')$ contains the set of 0-simplices of some (finite or infinite) directed geodesic of type 2-4 in F (black dots in Figure 3.2). If $F \cap F'$ contains only 1 black dot the statement is obvious. As we assumed $F \cap F'$ is not contained in the set of 0-simplices of a directed geodesic of type 2-4, there is a vertex in $H_i \cap (F \cap F')$ different from the black ones. Moreover by Step 1 and the form of a directed geodesic in a flat (see Figure 3.1) there is a vertex in $H_i \cap (F \cap F')$ at distance at most 2 from some black vertex.

If one of the vertices marked by white dots in Figure 3.2 belongs to $F \cap F'$ (say $u \in F \cap F'$), then by Step 1 all the vertices marked by white dots adjacent to two black vertices belong to $F \cap F'$ (they are 0-simplices of a directed geodesic in F joining u with a properly chosen black vertex).

If one of the vertices marked by white squares belongs to $F \cap F'$ (say $v \in F \cap F'$), then either we can connect v with a black vertex by a convex geodesic passing through a vertex marked by a white dot and proceed as above or $F \cap F'$ contains only two black vertices (a and b).

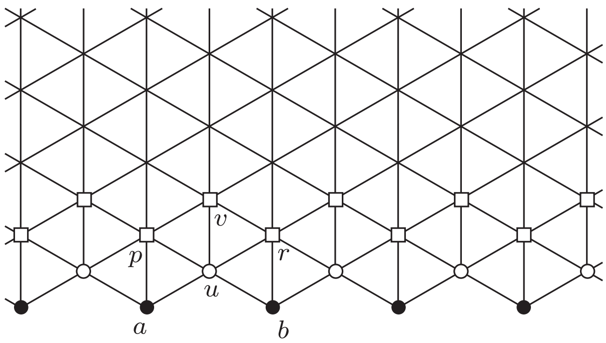

Figure 3.2.

In the latter case by Step 1 and the maximality of $F \cap F'$, vertices a, b and v are pairwise connected by geodesics of type 2-4 and length 2 in F' as well as in F. Thus a vertex $u' \in N(a) \cap N(b) \cap N(v) \subset F'$ is connected by edges with $a, b, v \in F$, so by Fact 2.2 it is connected to every vertex of $N(u) \subset F$, contradicting the maximality of $F \cap F'$ (we obtain a flat $\bar{F}$, equivalent to F, by replacing u with u').

Step 3: $\partial(H_1 \cap \ldots \cap H_{12}) \subset F \cap F'$.

Let $a_i \in \partial H_i$ and $a_{i+1} \in \partial H_{i+1}$. One of the half-planes H_i and H_{i+1} is of type 3-3, the other is of type 2-4 and 'the Euclidean angle' between them is $\frac{5\pi}{6}$. The directed geodesic in F either from a_i to a_{i+1} or from a_{i+1} to a_i passes through $\partial H_i \cap \partial H_{i+1}$ (compare Figure 3.1), so by Steps 1 and 2 we have $\partial H_i \cap \partial H_{i+1} \subset F \cap F'$.

If $H_i \neq \mathbb{E}^2_\triangle$ and $H_{i+1} = \mathbb{E}^2_\triangle$, then there are arbitrarily large $H \in \mathcal{H}_{i+1}$ such that $\partial H \cap (F \cap F') \neq \emptyset$, thus by the same argument as above $\partial H_i \cap \partial H \subset F \cap F'$ and an infinite ray in ∂H_i is contained in $F \cap F'$. This completes the proof of Step 3.

Step 4: $F \cap F' = H_1 \cap \ldots \cap H_{12}$.

The inclusion $\subset$ follows directly from the definition of H_i. If $H_1 \cap \ldots \cap H_{12}$ has no internal vertices, then the opposite inclusion is an immediate consequence of Step 3. Otherwise $\partial(H_1 \cap \ldots \cap H_{12})$ disconnects F into two connected components, one of which is spanned by the internal vertices of $H_1 \cap \ldots \cap H_{12}$. Choose any vertex v in this component and a convex geodesic g in F passing through v with endpoints on $\partial(H_1 \cap \ldots \cap H_{12})$. By Steps 1 and 3 $g \subset F \cap F'$. $\square$

Corollary 3.14. *For any two flats F, F' in a systolic complex X there is an isometry $\varphi : F' \to F$ extending $id_{F \cap F'}$.*

Proof: We need to show that if $A = H_1 \cap \ldots \cap H_{12}$ and $A' = H'_1 \cap \ldots \cap H'_{12}$ (where H_i and H'_i are as in Theorem 3.13 and H_i and H'_i are of the same type) are isometric subcomplexes of $\mathbb{E}^2_\triangle$, then this isometry can be extended to an automorphism of $\mathbb{E}^2_\triangle$. This follows from the fact, that A is uniquely determined up to an automorphism of $\mathbb{E}^2_\triangle$ by the sequence of integers $(s_1, \ldots, s_{12})$, where $s_i := |A^{(0)} \cap \partial H_i|$. $\square$

4. Asymptotic cones

In this section we prove that any geodesic in an asymptotic cone of $X^{(1)}$ (where X is a systolic complex) can be obtained as an ultralimit of geodesics in X. This fact will be used in the proof of Theorem 5.13.

Throughout the paper we only consider asymptotic cones of the 1-skeleta of systolic complexes, thus we will omit words 'the 1-skeleton' for brevity and refer to them as to 'asymptotic cones of systolic complexes'.

Below we shortly recall the definition of an asymptotic cone. For a more detailed introduction to asymptotic cones we refer the reader to [BH] and [DS].

Let ω be a non-principal ultrafilter over $\mathbb{N}$, i.e. a finitely additive measure on the class $\mathcal{P}(\mathbb{N})$ of all subsets of $\mathbb{N}$ such that each subset has measure 0 or 1, all finite sets have measure 0 and $\mathbb{N}$ has measure 1. We say that $g \in \mathbb{R}$ is an ω-limit of a sequence $(x_n)_{n=1}^{\infty}$ of real numbers (and denote it $g = \lim_\omega x_n$) if every neighbourhood of g contains elements x_n for ω-almost all indices n. We also write $\lim_\omega x_n = \infty$ if for every M the inequality $x_n > M$ holds for ω-almost all indices n. The reason for considering ω-limits is that every sequence of non-negative real numbers has a unique (finite or infinite) ω-limit.

Definition 4.1. *Let (X_n, d_n), $n \in \mathbb{N}$ be a sequence of metric spaces, ω a non-principal ultrafilter and $\star_n \in X_n$ a sequence of points. The metric space*

$$\lim_\omega (X_n, d_n)_{\star_n} = \big\{[(x_n)]_\sim :\ x_n \in X_n,\ d\big([(x_n)]_\sim, [(\star_n)]_\sim\big) < \infty\big\}$$

where $(x_n) \sim (y_n)$ if and only if $\lim_\omega d_n(x_n, y_n) = 0$, with the metric d defined by

$$d\big([(x_n)]_\sim, [(y_n)]_\sim\big) = \lim_\omega d(x_n, y_n)$$

is the ω-limit of the sequence (X_n, d_n) *relative to the observation point $(\star_n)$.*

Definition 4.2. *Let (X, d) be a metric space, ω a non-principal ultrafilter, $\lambda = (\lambda_n)_{n=1}^{\infty}$ a sequence of positive numbers such that $\lim_{n\to\infty} \lambda_n = \infty$ and $\star = (\star_n)_{n=1}^{\infty}$ a sequence of points in X. Then the metric space*

$$Cone_\omega(X; \lambda, \star) = \lim_\omega (X, \tfrac{1}{\lambda_n} d)_{\star_n}$$

is the asymptotic cone of X *relative to the scaling sequence λ and the observation point $\star$.*

If A_n is a sequence of subsets of X, we treat $\lim_\omega A_n$ as a subset of $\mathrm{Cone}_\omega(X; \lambda, \star)$. Formally this should be written as $\lim_\omega (A_n, \lambda_n^{-1} d_X)$, where d_X is the restricted metric, but we will omit metrics, for brevity. The other convention will be to write '(x_n)' instead of '$[(x_n)]_\sim$' for points of asymptotic cones, what should not cause any misunderstandings.

As asymptotic cones of (L, C)-quasi isometric spaces are L-bi-Lipschitz equivalent, we can define an asymptotic cone of a group, considered up to bi-Lipschitz equivalence. Notice, that since a group treated as a metric space is homogeneous, an asymptotic cone of a group does not depend on an observation point.

An asymptotic cone of a geodesic space is a geodesic space, since if (x_n) and (y_n) represent points of $\mathrm{Cone}_\omega(X; \lambda, \star)$ and g_n is a geodesic in X joining x_n with y_n, $n = 1, 2, \ldots$, then $\lim_\omega g_n$ is a geodesic in $\mathrm{Cone}_\omega(X; \lambda, \star)$ joining (x_n) with (y_n). However, there can exist geodesics in $\mathrm{Cone}_\omega(X; \lambda, \star)$ which do not arise in this way. The aim of this section is to show this is not the case of systolic spaces (Proposition 4.4). The first step in the proof is the following lemma.

Lemma 4.3. *If x_0, x_1, x_2 are vertices of a systolic complex X, then there exist geodesics γ_i with endpoints x_{i-1} and x_{i+1} for $i = 0, 1, 2$ (we use the cyclic order of indices) such that:*

(1) *$\gamma_i \cap \gamma_{i+1}$ is a geodesic (possibly degenerated) with endpoints x_{i-1} and x'_{i-1};*

(2) *if we denote the subgeodesic (possibly degenerated) with endpoints x'_{i-1} and x'_{i+1} by $\gamma'_i \subset \gamma_i$, then either $x'_0 = x'_1 = x'_2$ or a minimal surface $S : \Delta \to X$ spanning the cycle $\gamma'_0 * \gamma'_1 * \gamma'_2$ has an equilaterally triangulated equilateral triangle as the domain.*

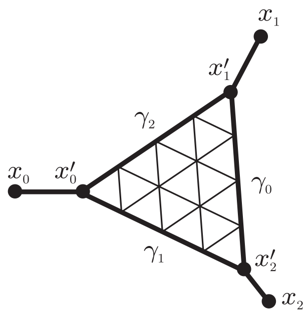

Figure 4.1.

Proof: There clearly exist in X geodesics satisfying (1). Take γ_0, γ_1, γ_2 such that the domain Δ of a minimal surface $S : \Delta \to X$ spanning the cycle $\gamma'_0 * \gamma'_1 * \gamma'_2$ has the minimal area.

Denote by $s_0, s_1, s_2 \in \Delta$ the vertices that are mapped to x'_0, x'_1 and x'_2, respectively. If $v \in \partial\Delta$ is different from s_0, s_1, s_2, then it has a nonpositive defect ($\mathrm{def}(v) \neq 2$ by the geodesity of γ'_i and $\mathrm{def}(v) \neq 1$ by the minimality of the area of Δ). Thus by the Gauss-Bonnet Lemma $\mathrm{def}(s_i) = 2$, $i = 0, 1, 2$ and the remaining vertices of Δ have defects 0. Therefore Δ is an equilaterally triangulated equilateral triangle. This is proved by induction on the perimeter of Δ – we cut out triangles touching one side of the triangle and apply the inductive assumption.

Notice that by Theorem 4.13 in [E] and by the fact that γ_i were chosen so that the area of Δ is minimal, the map S is an isometric embedding of $\Delta^{(1)}$. $\square$

Proposition 4.4. *Let $Cone_\omega(X; \lambda, \star)$ be an asymptotic cone of a systolic complex X. Then for every geodesic $g : [0, l] \to Cone_\omega(X; \lambda, \star)$ there exists a sequence of geodesics $g_n : [0, s_n] \to X$, $n = 1, 2, \dots$ such that $g = \lim_\omega g_n$. Moreover, if $(x_n) = g(0)$ and $(z_n) = g(l)$, then g_n can be chosen to have endpoints at x_n and z_n, for $n = 1, 2, \dots$*

Proof: Let $g(0) = x = (x_n)$, $g(\frac{l}{2}) = y = (y_n)$, $g(l) = z = (z_n)$, where x_n, y_n, z_n are vertices of X. By Lemma 4.3 there are geodesics α_n, β_n, γ_n connecting y_n with z_n, z_n with x_n and x_n with y_n, respectively, and vertices $x'_n \in \beta_n \cap \gamma_n$, $y'_n \in \gamma_n \cap \alpha_n$, $z'_n \in \alpha_n \cap \beta_n$ such that:

$$
\begin{aligned}
d(x_n, y_n) &= d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n) = a_n + t_n + b_n \\
d(y_n, z_n) &= d(y_n, y'_n) + d(y'_n, z'_n) + d(z'_n, z_n) = b_n + t_n + c_n \\
d(z_n, x_n) &= d(z_n, z'_n) + d(z'_n, x'_n) + d(x'_n, x_n) = c_n + t_n + a_n
\end{aligned}
$$

where $t_n = d(x'_n, y'_n) = d(y'_n, z'_n) = d(z'_n, x'_n)$ is the length of the side of the equilateral triangle and $a_n = d(x_n, x'_n)$, $b_n = d(y_n, y'_n)$, $c_n = d(z_n, z'_n)$. Hence

$$
\lim_\omega \lambda_n^{-1}(a_n + t_n + b_n) = d(x, y) = \frac{1}{2} l = d(y, z) = \lim_\omega \lambda_n^{-1}(b_n + t_n + c_n)
$$

and

$$\lim_\omega \lambda_n^{-1}(a_n + t_n + c_n) = d(x,z) = l$$

what implies (as $t_n, b_n \geq 0$):

$$\lim_\omega \lambda_n^{-1} b_n = \lim_\omega \lambda_n^{-1} t_n = 0.$$

Thus the sequences (y_n), (y_n') and (x_n') represent the same point $y \in \mathrm{Cone}_\omega(X;\lambda,\star)$. By putting $g_n^1 := \beta_n$ we obtain a sequence of geodesics in X such that $\lim_\omega g_n^1$ has endpoints $g(0)$ and $g(l)$ and passes through $g(\frac{l}{2})$.

Iterating the above procedure we construct a double-indexed sequence of geodesics $g_n^k : [0, s_n] \to X$ satisfying:

(1) $g_n^k(0) = x_n$ and $g_n^k(s_n) = z_n$,
(2) $g_n^k(s_n \cdot \frac{m}{2^k}) = g_n^{k'}(s_n \cdot \frac{m}{2^k})$ for $k' \geq k$ and $m = 0, 1, \ldots, 2^k$,
(3) $\lim_\omega g_n^k$ passes through $g(l \cdot \frac{m}{2^k})$, $m = 0, 1, \ldots, 2^k$.

Now consider the diagonal subsequence of geodesics $h_n = g_n^n$. Since for a fixed k and $0 \leq m \leq 2^k$ we have $h_n(s_n \cdot \frac{m}{2^k}) = g_n^k(s_n \cdot \frac{m}{2^k})$ for almost all indices n, the sequence $(h_n(s_n \cdot \frac{m}{2^k}))_{n=1}^\infty$ represents $g(l \cdot \frac{m}{2^k}) \in \mathrm{Cone}_\omega(X;\lambda,\star)$, so $h = \lim_\omega h_n$ passes through $g(l \cdot \frac{m}{2^k})$. Thus geodesics g and h (h is a geodesic as an ultralimit of geodesics) coincide on a dense set, so $g = h = \lim_\omega h_n$ and g is an ultralimit of geodesics in X. □

Proposition 4.5. *Let F_n be a sequence of flats in a systolic complex X. Then the ultralimit $\lim_\omega F_n \subset \mathrm{Cone}_\omega(X;\lambda,\star)$ is isometric to a plane P with the following metric:*

$$P = \{(x,y,z) \in (\mathbb{R}^3, \|\cdot\|_\infty) : \ x + y + z = 0\}.$$

Proof: As flats are isometric to $\mathbb{E}^2_\triangle$, we only need to prove that $P = \mathrm{Cone}_\omega(\mathbb{E}^2_\triangle;\lambda,\star)$. The statement follows from two facts: first that the 0-skeleton of $\mathbb{E}^2_\triangle$ (with the metric induced from the 1-skeleton of $\mathbb{E}^2_\triangle$) is isometric to

$$P_0 = \{(x,y,z) \in (\mathbb{Z}^3, \|\cdot\|_\infty) : \ x + y + z = 0\},$$

and the second that the following map is an isometry (where $\star = (\star_1, \star_2, \star_3)$):

$$\mathrm{Cone}_\omega(P_0;\lambda,\star) \ni \big((x_n, y_n, z_n)\big)_{n=1}^\infty \mapsto \Big(\lim_\omega \frac{x_n - \star_1}{\lambda_n}, \lim_\omega \frac{y_n - \star_2}{\lambda_n}, \lim_\omega \frac{z_n - \star_3}{\lambda_n}\Big) \in P$$

5. Isolated Flats Property

In a systolic complex there are no flats of dimension higher than 2, so we need to modify the Isolated Flats Property presented in [Hr2] for CAT(0)-spaces in the following way:

Definition 5.1. (Isolated Flats Property) *A cocompact systolic complex X has the* Isolated Flats Property *if there exists a function $\psi : \mathbb{N} \to \mathbb{N}$ such that*

$$\operatorname{diam}(\mathcal{N}_c(F) \cap \mathcal{N}_c(F')) \leq \psi(c), \text{ for any non-equivalent flats } F, F' \subset X. \tag{5.1}$$

One of the first consequences of the Isolated Flats Property of a systolic complex X is that groups acting cocompactly and properly discontinuously on X satisfy the so-called $\mathbb{Z} \times \mathbb{Z}$-conjecture (see Corollary 5.4). This fact will be used in Section 6 to prove that such groups are relatively hyperbolic with respect to their maximal virtually abelian rank 2 subgroups.

Non-trivial examples of systolic complexes with the Isolated Flats Property were constructed by T. Januszkiewicz and J. Świątkowski, using the technique of developments of billiards introduced in [JS3]. In particular, they constructed a group with one end, acting cocompactly and properly discontinuously on some systolic normal pseudomanifold with the Isolated Flats Property.

Fact 5.2. *If X is a locally finite systolic complex with the Isolated Flats Property, then every compact subcomplex of X intersects flats from only finitely many equivalence classes.*

Proof: It suffices to consider the case when the subcomplex is a single vertex v. By (5.1) non-equivalent flats passing through v have distinct intersections with the ball $\mathcal{N}_{\psi(0)}(v)$, which by the local finiteness of X has only finitely many subcomplexes. □

Proposition 5.3. *Let X be a systolic complex with the Isolated Flats Property and G a group acting cocompactly and properly discontinuously on X. Then for every flat F and the stabilizer* $\operatorname{Stab}(Th(F))$ *of its thickening holds:*

(1) $\operatorname{Stab}(Th(F))$ *acts cocompactly on $Th(F)$,*

(2) $\operatorname{Stab}(Th(F))$ *is a maximal virtually abelian rank 2 subgroup of G.*

Proof: Suppose there are vertices $v_i \in Th(F)$, $i = 1, 2, \ldots$ representing pairwise different orbits of the action of $\operatorname{Stab}(Th(F))$. By the local finiteness of X and by Theorem 3.2 we may assume that all v_i lie in the same flat F' (equivalent to F). As the number of G-orbits in $X^{(0)}$ is finite, we can choose a subsequence $v_{k_i} \in F'$, $i = 0, 1, 2, \ldots$ of vertices from the same G-orbit, so there exist elements $g_i \in G$, such that $g_i(v_{k_0}) = v_{k_i}$. Hence flats $g_i^{-1}(F')$, $i = 0, 1, 2, \ldots$ pass through v_{k_0}, so by Fact 5.2 there are two equivalent among them. Therefore $g_i g_j^{-1} \in \operatorname{Stab}(Th(F))$ for some $i \neq j$, contradicting the assumption that v_{k_i} and v_{k_j} were chosen from distinct $\operatorname{Stab}(Th(F))$-orbits.

As the stabilizer acts cocompactly and properly discontinuously on $Th(F)$, the induced action of $\operatorname{Stab}(Th(F))$ on $\mathbb{E}^2_\triangle$ (see Theorem 3.2) is such by Corollary 5.5 in [E]. As the group of automorphisms of $\mathbb{E}^2_\triangle$ is virtually abelian rank 2, the stabilizer $\operatorname{Stab}(Th(F))$ is also such and by Theorem 3.4 it has no finite extension in G, i.e. it is a maximal virtually abelian rank 2 subgroup. □

Corollary 5.4. *Let G be a group acting cocompactly and properly discontinuously on a systolic complex X with the Isolated Flats Property. Then there is a bijective correspondence between equivalence classes of flats in X and maximal virtually abelian rank*

2 subgroups of G, established by the map assigning to each flat F the stabilizer of its thickening $\mathrm{Stab}(Th(F))$.

Proof: It is an immediate consequence of Theorem 3.4 and Proposition 5.3. $\square$

5.1. Triplanes

This section is devoted to the proof of the theorem stating that a cocompact systolic complex satisfies the Isolated Flats Property if and only if it does not contain isometrically embedded triplanes. This is a simplicial analog of the CAT(0)-result of D. Wise (see [Hr1]). A triplane in Wise's theorem is the geodesic metric space obtained by gluing three half-planes by isometries along their boundaries. In the systolic case there are three similar configurations:

(a) By gluing three half-planes of type 3-3 (see Definition 2.9) by isomorphisms along their boundaries we obtain the *triplane of type 3-3* (Figure 5.1(a)).

(b) Let H_i be a half-plane of type 3-3 and $l_i \subset H_i$ the convex geodesic at distance 1 from ∂H_i, for $i = 0, 1, 2$. The *twisted triplane of type 3-3* is constructed by identifying ∂H_i with $l_{i+1} \subset H_{i+1}$ (as in Figure 5.1(b)) and taking the flag completion of the obtained complex (the group of its automorphisms is generated by a single glide rotation).

(c) To obtain the *triplane of type 2-4* (Figure 5.1(c)), we take a sequence of tetrahedra $\sigma_i = \sigma(v_i, a_i, b_i, c_i)$ and $\sigma_i' = \sigma(v_{i-1}, a_i, b_i, c_i)$, for $i \in \mathbb{Z}$ and glue half-planes of type 2-4 (see Definition 2.9) along the geodesics α, β, γ, spanned by the sets of vertices $\{v_i, a_i : i \in \mathbb{Z}\}$, $\{v_i, b_i : i \in \mathbb{Z}\}$ and $\{v_i, c_i : i \in \mathbb{Z}\}$, respectively, so that the positive boundary vertices are glued to v_i.

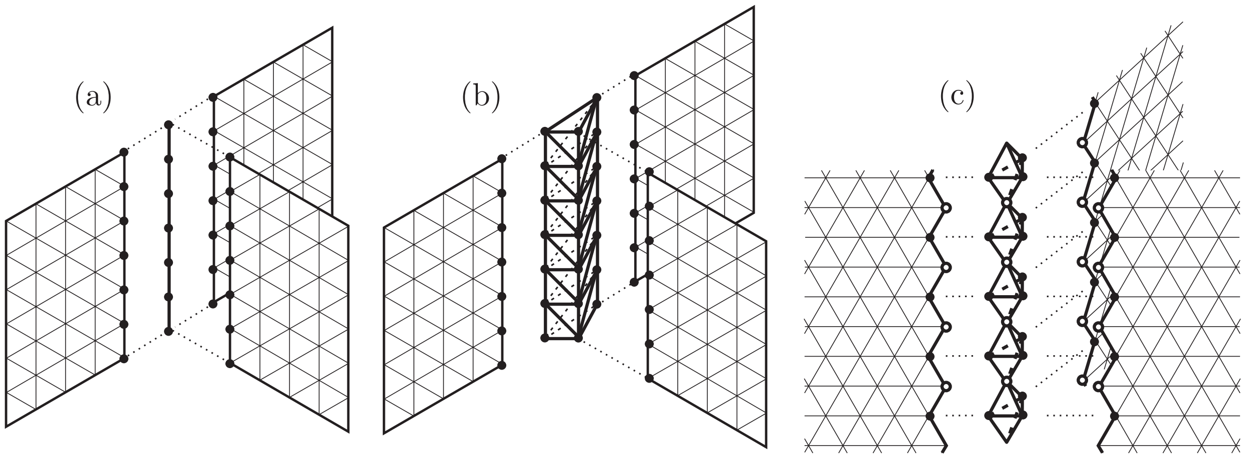

Figure 5.1.

In every triplane there are 3 distinct embedded flats F_0, F_1, F_2 having the property that any two of them span the whole triplane.

Triplanes are systolic complexes. To prove this it suffices to check that links at their vertices are 6-large. It is immediate for vertices in the interiors of the glued half-planes. Links at other vertices are presented in Figure 5.2.

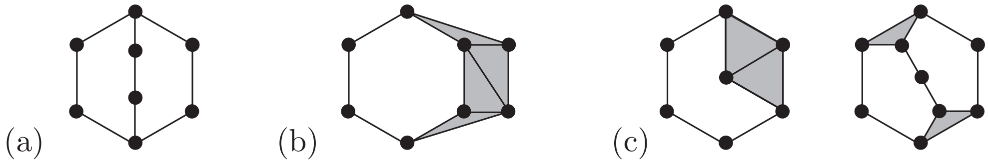

Figure 5.2.

Proposition 5.5. *Let T be a triplane, X a systolic complex and $f : T \to X$ a simplicial map such that the restriction of f to the 1-skeleton of $N(v)$ is an isometric embedding for any vertex $v \in T$ and $\operatorname{diam}(\operatorname{Im} f) \geq 3$. Then f restricted to the 1-skeleton of T is an isometric embedding.*

Proof: Since any two vertices $v, w \in T$ are contained in some flat F_i which is a geodesic subcomplex T, it suffices to prove that $f|_{F_i}$ is an isometric embedding for $i = 0, 1, 2$. All restrictions $f|_{F_i}$ are locally isometric immersions, thus to prove that they are isometric embeddings we only need to prove that their images have diameters at least 3 (Theorem 3.1). Since there are two points $x, y \in T$ such that $d(f(x), f(y)) \geq 3$, one of the flats has the image of diameter at least 3, hence it is an isometric embedding and all flats have images of infinite diameter. □

Lemma 5.6. *Let X be a locally finite cocompact systolic complex and $v \in \mathbb{E}^2_\triangle$ an arbitrary vertex.*

(1) *For any $r > 0$ there exists $a = a(r) > r$ such that if $\varphi : \mathcal{N}_a(v) \to X$ is an isometric embedding, then $\varphi(\mathcal{N}_r(v))$ is contained in some flat F_φ.*
(2) *If X satisfies the Isolated Flats Property, then there exists $r_0 = r_0(X)$ such that for any $r \geq r_0$ the flat F_φ defined in (1) is unique up to flat equivalence.*

Proof: a.c. Suppose there exist a sequence of integers $a_k > 0$ and a sequence of isometric embeddings $\varphi_k : \mathcal{N}_{a_k}(v) \to X$ such that $\lim a_k = \infty$ and $\varphi_k(\mathcal{N}_r(v))$ is not contained in any flat, for $k = 1, 2, \ldots$. By cocompactness of X we may assume (passing to a subsequence) that vertices $\varphi_k(v) \in X$ coincide. Thus, as X is locally finite, we can choose a subsequence a_{k_i}, $i = 1, 2, \ldots$ such that $\varphi_{k_i}|_{\mathcal{N}_r(v)}$ coincide. By the standard diagonal argument we find a subsequence $\varphi_{k_{i_j}}$ convergent to a flat F. Thus $\varphi_{k_{i_j}}(\mathcal{N}_r(v)) \subset F$, contradicting the assumption.

The remaining part of the lemma follows from (5.1), by putting $r_0 = \psi(0)$. □

Theorem 5.7. *A cocompact systolic space X has the Isolated Flats Property if and only if it does not contain isometrically embedded triplanes.*

Proof: The 'only if' part is trivial, as any triplane contains non-equivalent flats with infinite intersection. We prove the 'if' part, i.e. X not satisfying the Isolated Flats Property contains an isometrically embedded triplane.

Step 1: *There exist flats $F, F' \subset X$ at infinite Hausdorff distance and bi-infinite geodesics $l \subset F$, $l \subset F'$ both of type 3-3 or both of type 2-4 such that* $\operatorname{hdist}_X(l, l') < \infty$.

As X does not satisfy the Isolated Flats Property, there is a constant $c \geq 0$ and two sequences of flats: $(F_n)_{n=1}^{\infty}$ and $(F'_n)_{n=1}^{\infty}$ such that $\mathrm{hdist}_X(F_n, F'_n) = \infty$, for $n = 1, 2, \ldots$ (flats at finite Hausdorff distance are equivalent) and

$$\lim_{n\to\infty} \mathrm{diam}(\mathcal{N}_c(F_n) \cap \mathcal{N}_c(F'_n)) = \infty. \tag{5.2}$$

We may assume, not losing generality, that the intersection $F_n \cap F'_n$ is maximal among all pairs of flats from equivalences classes of F_n and F'_n.

Let c be minimal and consider the case $c = 0$. By Corollary 3.14 there exist isometries $\varphi_n : F_n \to F'_n$ extending $id_{F_n \cap F'_n}$. By Theorem 3.13 there exist (in the boundary of $F_n \cap F'_n$) geodesics $g_n \subset F_n \cap F'_n$ of type 3-3 or 2-4 such that $\lim |g_n| = \infty$ and the middle vertices $a_n \in g_n \subset F_n$ are such that $N(a_n) \cap F_n \neq N(a_n) \cap F'_n$.

Suppose the ball $\mathcal{N}_r(a_n) \cap F_n$, $r > 0$ is contained in $\frac{1}{100}r$-neighbourhood of F'_n. Then we join the vertices $p_1, \ldots, p_6$ of the hexagon $\mathcal{N}_r(a_n) \cap F_n$ with the closest vertices $p'_1, \ldots, p'_6$ of F'_n and connect p'_i with p'_{i+1} by allowable geodesics in F'_n, for $i = 1, \ldots, 6$. By the Fellow Traveller Property (Proposition 2.6) the minimal surfaces $S : \Delta \to X$ and $S' : \Delta' \to X$ obtained by restricting F and F' have the property that $S(\partial\Delta)$ and $S'(\partial\Delta')$ are $(\frac{3}{100}r + 1)$-close to each other. Then by Theorem 4.16 in [E] we can find a flat $\bar{F}_n$ equivalent to F_n, such that $N(a_n) \cap \bar{F}_n = N(a_n) \cap F'_n$, contadicting the maximality of $F_n \cap F'_n$).

Thus for any $r > 0$ the ball $\mathcal{N}_r(a_n) \cap F_n$ is not contained in $\frac{1}{100}r$-neighbourhood of F'_n and by the standard diagonal argument we obtain flats F and F' at infinite Hausdorff distance and bi-infinite geodesic $g \subset F \cap F'$ of type 3-3 or 2-4.

Now consider the case $c > 0$. By (5.2) there exist vertices $x_n, y_n \in F_n$ and $x'_n, y'_n \in F'_n$ such that:

$$d_X(x_n, x'_n) \leq 2c \quad \text{and} \quad d_X(y_n, y'_n) \leq 2c$$
$$\lim_{n\to\infty} d_X(x_n, y_n) = \infty \quad \text{and} \quad \lim_{n\to\infty} d_X(x'_n, y'_n) = \infty$$

Connect x_n with y_n and x'_n with y'_n by allowable geodesics α_n in F_n and α'_n in F'_n, respectively (they are allowable in flats, thus may be not allowable in X, but by Lemma 3.8 they are 1-close to allowable geodesics in X). By the Fellow Traveller Property the Hausdorff distance between α_n and α'_n does not exceed $3c + 3$. A directed geodesic in a flat consists of two subgeodesics (possibly degenerated): a directed geodesic of type 3-3 and a directed geodesic of type 2-4 (Figure 3.1), thus we may assume (by replacing α_n and α'_n with their subgeodesics and passing to subsequences) that all α_n are of the same type (3-3 or 2-4) and all α'_n are also such.

Denoting by $a_n \in \alpha_n$ the middle vertex, for $r > 100c$ the ball $\mathcal{N}_r(a_n) \cap F_n$ is not contained in $\frac{1}{100}r$-neighbourhood of F'_n (otherwise, proceeding similarly as in the case $c = 0$, we obtain contradiction with the minimality of c). Thus, by the standard diagonal argument, we obtain as the limit two flats F and F' and a bi-infinite geodesics of type 3-3 or 2-4: $g \subset F$ and $g' \subset F'$ such that $\mathrm{hdist}_X(g, g') < \infty$ and $\mathrm{hdist}_X(F, F') = \infty$.

By Proposition 3.11 a bi-infinite geodesic of type 3-3 in a flat is quasi-convex in the systolic complex X, whereas a bi-infinite geodesic of type 2-4 in a flat is not (its convex hull contains the whole flat). Moreover, by Proposition 3.12, bi-infinite geodesics in X at finite Hausdorff distance either are both quasi-convex or both are not. Thus g and g' are either both of type 3-3 or both are of type 2-4.

Step 2: *There exists a 'large-scale triplane', i.e. half-planes H_1, H_2, H_3 of the same type such that:*

$$\text{(5.3)}\quad \operatorname{hdist}_X(\mathcal{N}_c(H_i)\cap H_j, \partial H_j) < \infty \textit{ and } \operatorname{hdist}_X(\partial H_i, \partial H_j) < \infty, \textit{ for all } i \neq j,\ c \geq 0.$$

By Step 1 of the proof there exist flats F_1 and F_2 and bi-infinite geodesics $g_1 \subset F_1$ and $g_2 \subset F_2$ either both of type 3-3 or both of type 2-4 such that $\operatorname{hdist}_X(g_1, g_2) < \infty$ and $\operatorname{hdist}_X(F_1, F_2) = \infty$. Geodesics g_1 and g_2 divide F_1 and F_2 into 4 half-planes of the same type: H_1^+, H_1^-, H_2^+, H_2^-. We show that three of them satisfy (5.3).

Obviously $\operatorname{hdist}_X(\mathcal{N}_c(H_i^{\pm})\cap H_i^{\mp}, \partial H_i^{\mp}) < \infty$, for $i = 1, 2$ and $c \geq 0$, so we only need to consider pairs $H_i^{\pm}$ and $H_j^{\pm}$ for $i \neq j$. Suppose that not all pairs satisfy (5.3). Without loss of generality, we can assume there is a constant $c \geq 0$ such that

$$\operatorname{hdist}_X(\mathcal{N}_c(H_1^+)\cap H_2^+, \partial H_2^+) = \infty.$$

Then there is a sequence of vertices $x_n \in H_2^+$, such that

$$\text{(5.4)}\qquad \operatorname{dist}(x_n, \partial H_2^+) > n \quad \text{and} \quad \operatorname{dist}_X(x_n, H_1^+) \leq c.$$

Since ∂H_2^+ is an allowable geodesic (allowable in F_2) of type 3-3 or 2-4, we see that any vertex $v \in H_2^+$ at distance at most n from ∂H_2^+ lies on some allowable geodesic (allowable in F_2) joining x_n with ∂H_2^+ or joining ∂H_2^+ with x_n. As by Proposition 3.8 allowable geodesics in F_2 are 1-close to allowable geodesics in X, by the Fellow Traveller Property and (5.4) we obtain $H_2^+ \subset \mathcal{N}_{3c'+3}(H_1^+)$, where $c' = \max\{c, \operatorname{hdist}_X(\partial H_1^+, \partial H_2^+)\}$.

Since $\operatorname{hdist}_X(\partial H_1^+, \partial H_2^+) < \infty$, we can find a sequence of vertices $y_n \in H_1^+$ such that

$$\operatorname{dist}(y_n, \partial H_1^+) > n \quad \text{and} \quad \operatorname{dist}_X(y_n, H_2^+) \leq 3c' + 3$$

and repeat the argument from the previous paragraph to show that H_1^+ and H_2^+ are actually at finite Hausdorff distance.

Thus any pair $H_1^{\pm}$, $H_2^{\pm}$ either satisfies (5.3) or is at finite Hausdorff distance. As there is at most one pair of half-planes at finite Hausdorff distance among them, we can choose three half-planes satisfying (5.3).

Step 3: *There exist flats F_1, F_2, F_3 and half-planes of the same type H_1, H_2, H_3 satisfying* (5.3) *such that $H_i \cup H_j \subset F_k$.*

In Step 2 we have constructed H_1, H_2, H_3, satisfying (5.3). Fix $i \neq j$. Geodesics ∂H_i and ∂H_j are at finite Hausdorff distance. Thus there is a simplicial isomorphism $\varphi : \partial H_i \to \partial H_j$ such that $d_X(v, \varphi(v)) < c$ for every vertex $v \in \partial H_i$ and some $c > 0$. Join v with $\varphi(v)$ by a geodesic α_v in X and for every edge $vw \subset \partial H_i$ consider a minimal surface $f_{vw} : \Delta_{vw} \to X$ spanning the closed path $vw * \alpha_w * \varphi(w)\varphi(v) * \alpha_v^{-1}$. Gluing these maps with embeddings $H_i \subset X$ and $H_j \subset X$ we obtain a simplicial map $f : P \to X$, where P is the union of disjoint half-planes P_i and P_j of the same type (mapped to H_i and H_j, respectively) and a strip S obtained by gluing Δ_{vw} for all edges vw such that $\partial S = \partial P_i \cup \partial P_j$. By Lemma 3.4(2) in [E] $\Delta_{vw} \subset \mathcal{N}_{\frac{1}{6}(2c+2)}(\partial \Delta_{vw})$, so ∂P_i, ∂P_j and S are pairwise at finite Hausdorff distance in P.

Choose half-planes $P'_i \subset P_i$ and $P'_j \subset P_j$ with the property that $f(P'_i)$, $f(P'_j)$ and $f(S)$ have pairwise disjoint 1-neighbourhoods. Now we iterate the following procedure:

(a) if there is a cycle of length 3 in P not bounding a triangle, then we cut out its filling and glue a single triangle instead;

(b) if every cycle of length 3 bounds a triangle and there is a vertex $v \in P$ adjacent to 4 or 5 triangles, then we cut out the open star of v and glue a disc without internal vertices such that f can be modified over the new triangulation (this is possible since X is systolic);

(c) if there is a vertex $v \in P$ adjacent to 6 or more triangles such that $\partial N(v)$ can be filled in X with a disc without internal vertices, then we proceed as in (b).

As we modify P, we modify f. Since $f|_{P'_i}$ and $f|_{P'_j}$ are isometric embeddings and S is at finite Hausdorff distance from $\partial P'_i$, every compact subcomplex of X contains images of only finitely many vertices of P. Thus the procedure terminates in every ball in P.

As the limit we obtain a systolic triangulation P' of a plane and a locally isometric immersion $f : P' \to X$ (i.e. the restriction of f to the 1-skeleton of $N(v)$ is an isometric embedding for any vertex $v \in P'$). Applying the Gauss-Bonnet Lemma to large systolic discs $\Delta \subset P'$ whose boundaries are concatenations of 4 geodesics: two being subsegments of $\partial P'_1$ and $\partial P'_2$ and two having lengths at most $\mathrm{hdist}_{P'}(\partial P'_1, \partial P'_2)$, we see that P' has at most 6 vertices of negative defects (by Remark 3.7 the sum of defects along any side of the quadrilateral is at most 1 and the defect at any of its four vertices is at most 2).

Thus there exists a systolic disc $K = \mathcal{N}_r(v) \subset P'$ such that every vertex in $P' \setminus K$ is adjacent to 6 triangles. We may assume $v \in P' \setminus (P'_1 \cup P'_2)$ and $r > 2 \cdot \mathrm{hdist}_{P'}(\partial P'_1, \partial P'_2)$. We prove by induction on c that for every $r \geq c \geq 0$ the subcomplex $\mathcal{N}_c(P'_1) \setminus K$ is isomorphic to a half-plane (of type 3-3 or 2-4) with cut out a disc intersecting its boundary. Two half-planes (of the same type) H and H' with infinite intersection such that $\mathrm{hdist}(\partial H, H \cap H') < \infty$ have the property that the closure of $H \setminus H'$ is also a half-plane (of type 3-3 or 2-4). Thus $(\mathcal{N}_r(P'_1) \setminus K) \setminus (P'_2 \setminus K)$ is equal to a half-plane with a cut-out systolic disc. Gluing it with $P'_2 \setminus K$ we obtain a complex isomorphic to $\mathbb{E}^2_\triangle$ with a hole. By the Gauss-Bonnet Lemma (applied to the hole) P' is isomorphic to $\mathbb{E}^2_\triangle$, so by Theorem 3.1 f is a flat.

Step 4: *There exists a triplane isometrically embedded in X.*

Let F_1, F_2, F_3 be flats constructed in Step 3. Choose F_1 and F_2 to have maximal intersection among pairs of flats from equivalence classes $[F_1]$ and $[F_2]$. As $F_1 \cap F_2$ contains a half-plane (of type 3-3 or 2-4), by Theorem 3.12 it is equal to some half-plane.

The geodesics $\partial H_1 \subset F_2 \cap F_3$, $\partial H_2 \subset F_1 \cap F_3$ and $\partial H_3 \subset F_1 \cap F_2$ are pairwise at finite Hausdorff distance. Consider the complex T obtained by gluing the strips $S_i \subset F_i$, for $i = 1, 2, 3$, bounded by these geodesics.

Choose a vertex $v \in F_i$ which is contained in exactly one of the flats F_1, F_2, F_3 and two disjoint cycles γ and γ' in T representing the generator of $\pi_1(T)$, such that the Hausdorff distance between v and any of the cycles is larger than $\max\{|\gamma|, |\gamma'|\}$ and v is contained in the compact connected component T' of $T \setminus (\gamma \cup \gamma')$. Span minimal surfaces $f : \Delta \to X$ and $f' : \Delta' \to X$ on the cycles γ and γ', respectively, and glue maps f, f' with the embedding $T' \subset X$ to obtain a simplicial map from a triangulated sphere

$p : S \to X$. By Theorem 2.5 p can be extended to a simplicial map $P : B \to X$, where B is a triangulated ball with no internal vertices such that $\partial B = S$. Then the link B_v is a triangulated disc of perimeter 6 and (as F_i is isometrically embedded) P maps internal vertices of B_v to $F_j \cup F_k$, $i \neq j \neq k \neq i$, and the boundary ∂B_v to the cycle having no diagonals. By Lemma 4.13 in [E] there is a flat F_i', equivalent to F_i, obtained by replacing $v \in F_i$ with some vertex $v' \in F_j \cup F_k$, $i \neq j \neq k \neq i$.

Iterating this procedure (as flats are isometrically embedded it terminates in every compact subcomplex of X), we obtain flats F_1', F_2', F_3' such that every vertex of $F_1' \cup F_2' \cup F_3'$ is contained in at least two of the flats and (by the maximality of $F_1 \cap F_2$) $H_3 = F_1' \cap F_2' = F_1 \cap F_2$ which is a half-plane of type 3-3 or 2-4. In particular, the half-planes $H_1 = F_1' \setminus H_3$ and $H_2 = F_2' \setminus H_3$ are contained in F_3' and $\mathrm{hdist}_X(\partial H_1, \partial H_2) \leq 2$.

If the half-planes are of type 3-3 and $\mathrm{hdist}_X(\partial H_1, \partial H_2) = 2$, then $F_1' \cup F_2' \cup F_3'$ is the isometrically embedded triplane of type 3-3 (Figure 5.1(a)).

If the half-planes are of type 3-3 and $\mathrm{hdist}_X(\partial H_1, \partial H_2) = 1$, then $F_1' \cup F_2' \cup F_3'$ is the complex obtained by gluing the strip in Figure 5.3 by isomorphism along the boundary components and gluing three half-planes of type 3-3 along three horizontal lines. Thus by taking the flag completion we obtain the twisted triplane of type 3-3 (Figure 5.1(b)). To see this we need to show that there is only one possible gluing of the strip in the figure below.

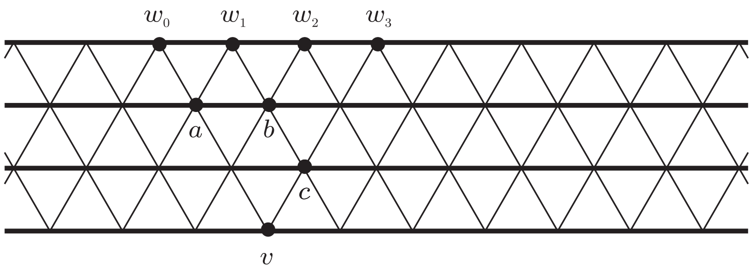

Figure 5.3.

Since the horizontal lines are geodesic in X, we have to identify the vertex v with the vertex w_i for some $i \in \{0, 1, 2, 3\}$. Actually, there are two cases: $i = 0$ and $i = 1$ (the other two are symmetric). In the case $i = 0$ the vertices a and v are connected by an edge, so by Fact 2.2 the quadrilateral has either the diagonal ac or the diagonal $bv = bw_0$. However, both cases are impossible as the endpoints of these diagonals are at distance 2 in one of the flats F_i' (which is isometrically embedded). In the case $i = 1$ we obtain a twisted triplane of type 3-3.

If the half-planes are of type 2-4, then the situation is as in Figure 5.4 (since every vertex of $F_1' \cup F_2' \cup F_3'$ belongs to at least two of the flats) – dark half-planes are H_2 and H_1 (contained in F_3') and the thick horizontal line is $\partial(F_1' \cap F_2')$. By Fact 2.2 the quadrilaterals $u_i a_i w_i a_{i-1}$ have the diagonals $w_i u_i$, thus $F_1' \cup F_2' \cup F_3'$ is a triplane of type 2-4 (Figure 5.1(c)).

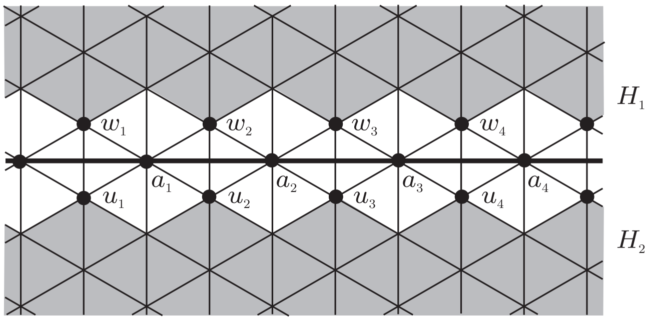

Figure 5.4.

All constructed triplanes are isometrically embedded, as flats F_1, F_2 and F_3 are such and any two vertices $v, w \in F_1 \cup F_2 \cup F_3$ can be connected by a geodesic contained in F_i for some i. $\square$

Motivated by the construction of triplanes, Piotr Przytycki presented the following short proof of the fact that a product of two finitely generated non-abelian free groups is a systolic group (this was an open question by now).

Proposition 5.8. The product $F_n \times F_m$, $n, m \geq 3$ of two finitely generated non-abelian free groups is a systolic group.

Proof: A finitely generated free group acts cocompactly and properly discontinuously on a regular tree T with all vertices of degree 3. Thus $F_n \times F_m$ acts cocompactly and properly discontinuously on $T \times T$ and the action preserves the product structure. The product $T \times T$ has a natural structure of a cubical complex. Let us triangulate each square as in Figure 5.5(a).

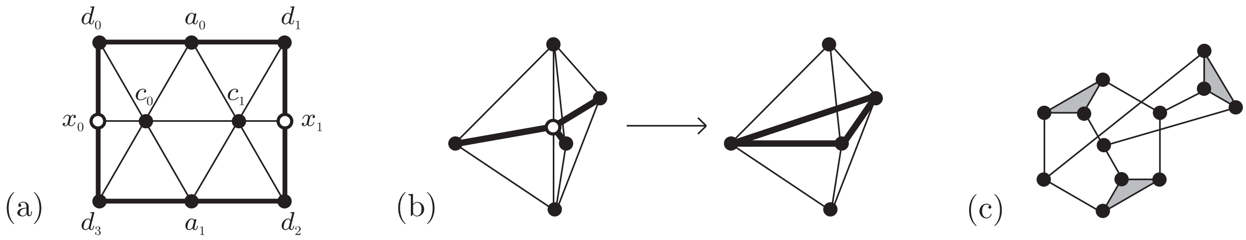

Figure 5.5.

The neighbourhood of every vertex x_i is the suspension of a tripod (see Figure 5.5(b)). We cut out every such subcomplex and glue the suspension of a triangle instead (as in Figure 5.5(b)). We obtain a 3-dimensional complex X. Notice, that any product of an infinite tripod in one component of $T \times T$ and a geodesic in the other component was replaced by a triplane either of type 3-3 or of type 2-4 (compare Figure 5.1).

Since the action of $F_n \times F_m$ on $T \times T$ preserves the cubical structure (and consequently simplicial structure) and preserves the product structure, it induces the cocompact and properly discontinuous simplicial action of $F_n \times F_m$ on X. Thus we only need to check that X is a systolic complex. Since X is obviously flag, it suffices to check that links at vertices of X are 6-large. The links at vertices c_i, $i = 0, 1$ and a_i, $i = 0, 1$ are shown in Figure

5.2(c) (first graph) and Figure 5.2(a), respectively. The links at vertices d_i, $i = 0, 1, 2, 3$ are shown in Figure 5.5(c). Thus all the links are 6-large. □

5.2. Relative hyperbolicity

There are three different approaches to relative hyperbolicity of a finitely generated group G with respect to a collection of its finitely generated subgroups $H_1, \ldots, H_n$. The first one was suggested by Brian Bowditch in [Bow] in terms of dynamics of properly discontinuous isometric actions of G on hyperbolic spaces. In [Fa] Benson Farb gave a definition in terms of coset graphs. It was proved by François Dahmani in [Da] that relative hyperbolicity in the sense of Bowditch implies relative hyperbolicity in the sense of Farb (called also *weak relative hyperbolicity*), but not conversely. However, Farb in [Fa] introduced an important additional condition called the Bounded Coset Property and it turns out ([Da]), that weak relative hyperbolicity with BCP is equivalent to relative hyperbolicity in the sense of Bowditch.

In this paper we use another approach, by Cornelia Druţu and Mark Sapir in [DS], in terms of asymptotic cones.

Definition 5.9. ([DS], Definition 1.10) *A complete geodesic metric space X is* tree-graded *with respect to a family $\mathcal{P}$ of its closed geodesic subsets (called* pieces*) if the following two properties are satisfied:*

(1) *Every two different pieces have at most one common point.*
(2) *Every simple geodesic triangle in X (i.e. a simple loop that is a concatenation of three geodesics) is contained in one piece.*

The restriction to simple geodesic triangles makes the condition (2) easier to check, but we obtain an equivalent definition by replacing the words 'simple geodesic triangle' with 'simple loop' ([DS], Proposition 2.15).

Definition 5.10. ([DS], Definition 3.19) *A metric space X is* asymptotically tree-graded *with respect to a family $\mathcal{A}$ of its subsets if for every (non-principal) ultrafilter ω, every sequence of scalars $\lambda = (\lambda_n)$ and every observation point $\star = (\star_n)$, the asymptotic cone $Cone_\omega(X; \lambda, \star)$ is tree-graded with respect to $\mathcal{A}_\omega = \{\lim_\omega A_n : \ A_n \in \mathcal{A}\}$.*

Notice that if $\lim_\omega \lambda_n^{-1} \operatorname{dist}(\star_n, A_n) = \infty$, then $\lim_\omega A_n = \emptyset$, so $\mathcal{A}_\omega$ contains the empty set. However, adding or removing the empty piece does not influence the property of being tree-graded.

Theorem 5.11. ([DS], Theorem 8.5) *A finitely generated group G is relatively hyperbolic with respect to a collection of its subgroups $H_1, \ldots, H_n$ if and only if G is asymptotically tree-graded with respect to the collection of cosets gH_i, where $g \in G$, $i = 1, \ldots, n$.*

We use the above characterization to prove that systolic groups acting geometrically on complexes with the Isolated Flats Property are relatively hyperbolic with respect to their maximal virtually abelian rank 2 subgroups (Corollary 5.14). We will need the following technical lemma:

Lemma 5.12. ([E], Lemma 4.15(1)) *Let Δ be a systolic triangulation of a 2-disc and $\gamma \subset \partial\Delta$ a geodesic in Δ. Denote by $\Delta' \subset \Delta$ the subcomplex obtained by cutting out open stars at every vertex $v \in \gamma$. Then $\mathrm{hdist}_\Delta(\Delta, \Delta') = 1$ and Δ' either has a disconnecting vertex or it is a systolic disc such that $\gamma' = \partial\Delta' \setminus \partial\Delta$ is a geodesic in Δ'.*

Theorem 5.13. *Let X be a locally finite cocompact systolic complex with the Isolated Flats Property and $\mathcal{F}$ the family of all flats in X. Then for every (non-principal) ultrafilter ω, a sequence of scaling constants $\lambda = (\lambda_n)$ and an observation point $\star = (\star_n)$ the asymptotic cone $Cone_\omega(X; \lambda, \star)$ is tree-graded with respect to $\mathcal{F}_\omega = \{\lim_\omega F_n,\ F_n \in \mathcal{F}\}$.*

Proof:

Step 1: $|F \cap F'| \leq 1$ *for any distinct flats $F, F' \in \mathcal{F}_\omega$.*

Suppose there exist distinct flats $F, F' \in \mathcal{F}_\omega$ and distinct points $x, y \in F \cap F'$. Then $F = \lim_\omega F_n$, $F' = \lim_\omega F'_n$ for some flats $F_n, F'_n \in \mathcal{F}$ and $x = (x_n)$, $y = (y_n)$, where x_n and y_n can be taken in F_n. Choose vertices $x'_n, y'_n \in F'_n$ such that $d(x_n, x'_n)$ and $d(y_n, y'_n)$ are minimal. Connect x_n with y_n and x'_n with y'_n by allowable geodesics $\gamma_n \subset F_n$ and $\gamma'_n \subset F'_n$, respectively. Join x_n with x'_n and y_n with y'_n by arbitrary geodesics α_n and β_n. Notice that γ_n and γ'_n are of length $O(\lambda_n)$ and α_n, β_n are of length $o(\lambda_n)$.

Let $c = a(r_0) + 1$ where $a(r_0)$ is the constant from Lemma 5.6. By the Isolated Flats Property $\mathrm{diam}(\mathcal{N}_c(F_n) \cap \mathcal{N}_c(F'_n))$ is bounded, thus we may assume (by replacing γ_n and γ'_n with their subgeodesics) that $\mathcal{N}_c(\gamma_n) \cap \mathcal{N}_c(\gamma'_n) = \emptyset$.

Let $S_n : \Delta_n \to X$ be a minimal surface spanning the closed path $\alpha_n * \gamma'_n * \beta_n^{-1} * \gamma_n^{-1}$. By the Gauss-Bonnet Lemma there are at most 6 internal vertices of negative defect in Δ_n (the sum of defects along each side of the quadrilateral is by Remark 3.7 at most 1 and the defect at any of the four vertices of the quadrilateral is at most 2). Thus by the Fellow Traveller Property and Proposition 3.8 we may assume (by replacing γ_n and γ'_n with their subgeodesics) that there are no internal vertices of negative defects in Δ_n.

We iterate c times the following procedure:

(∗) Delete all triangles adjacent to α_n. Then cut the resulting complex at every disconnecting vertex (if there are such) and take the component having the largest intersection with $\partial\Delta_n$.

We can prove by induction on c (using Lemma 5.12) that the obtained complex is connected and is at $o(\lambda_n)$-Hausdorff distance from Δ_n (as α_n and β_n are at distance larger than 2c, if we obtain a complex with a disconnecting vertex in some step of the procedure, all components but one have perimeter $o(\lambda_n)$, thus by Lemma 3.4 in [E] they have diameter $o(\lambda_n)$).

We apply the procedure (∗) iterated c times for β_n, γ_n and γ'_n, instead of α_n, finally obtaining a connected complex Δ'_n, such that $\mathrm{hdist}_{\Delta_n}(\Delta_n, \Delta'_n) = o(\lambda_n)$ and the length of $\mathcal{N}_c(\Delta'_n) \cap \partial\Delta_n$ is $O(\lambda_n)$.

Thus for every vertex $v \in \Delta'_n$ the ball $\mathcal{N}_c(v) \subset \Delta_n$ is an equilaterally triangulated regular hexagon, so by Lemma 5.6 $S_n(\mathcal{N}_c(v))$ is contained in some flat $\bar{F}^v_n$. Since Δ'_n is connected, the flat $\bar{F}_n^v$ does not depend on $v \in \Delta_n'$ (Lemma 5.6), so $S_n(\Delta_n')$ is contained in some flat $\bar{F}_n$. Therefore $\operatorname{diam}(\mathcal{N}_c(\bar{F}_n) \cap \mathcal{N}_c(F_n)) = O(\lambda_n)$, contradicting the Isolated Flats Property.

Step 2: *Every simple geodesic triangle in* $\mathrm{Cone}_\omega(X; \lambda, \star)$ *is contained in some flat* $F \in \mathcal{F}_\omega$.

Let $a = (a_n)$, $b = (b_n)$, $c = (c_n)$ be vertices of a simple geodesic triangle in the asymptotic cone $\mathrm{Cone}_\omega(X; \lambda, \star)$ and α, β, γ its sides opposite to a, b and c, respectively. By Proposition 4.4 there are sequences of geodesics α_n, β_n, γ_n in X such that $\lim_\omega \alpha_n = \alpha$, $\lim_\omega \beta_n = \beta$, $\lim_\omega \gamma_n = \gamma$. As the ω-limit of the closed path $\alpha_n \cup \beta_n \cup \gamma_n$ is a simple loop, we can assume (not losing generality) that $\alpha_n \cup \beta_n \cup \gamma_n$ is a cycle in X, for $i = 1, 2, \ldots$. By the same argument there exists a sequence of integers $\rho_n = o(\lambda_n)$ such that the geodesics

$$\begin{aligned} \alpha_n' &= \alpha_n \setminus (\mathcal{N}_{\rho_n}(b_n) \cup \mathcal{N}_{\rho_n}(c_n)) \\ \beta_n' &= \beta_n \setminus (\mathcal{N}_{\rho_n}(c_n) \cup \mathcal{N}_{\rho_n}(a_n)) \\ \gamma_n' &= \gamma_n \setminus (\mathcal{N}_{\rho_n}(a_n) \cup \mathcal{N}_{\rho_n}(b_n)) \end{aligned}$$

have pairwise disjoint $16c$-neighbourhoods for $c = a(r_0) + 1$, where $a(r_0)$ is the constant from Lemma 5.6.

Denote by $S_n : \Delta_n \to X$ a minimal surface spanning the cycle $\alpha_n \cup \beta_n \cup \gamma_n$ (we treat $\alpha_n \cup \beta_n \cup \gamma_n$ as the boundary of Δ_n, what should not cause any misunderstandings). Then Δ_n is a systolic complex (Proposition 2.4) and by the Gauss-Bonnet Lemma and Remark 3.7 there are at most 3 internal vertices in Δ_n of negative defects. Denote them by v_n^1, v_n^2, v_n^3. The sequence ρ_n can be chosen so that

$$\max\{d_{\Delta_n}(v_n^i, a_n), d_{\Delta_n}(v_n^i, b_n), d_{\Delta_n}(v_n^i, c_n) : i = 1, 2, 3\} \notin [\rho_n, \rho_n + 80c]$$

As by Corollary 4.10 in [HS] balls about vertices in systolic complexes are geodesically convex, the complex

$$\Delta_n' = \Delta_n \setminus (\mathcal{N}_{\rho_n}(a_n) \cup \mathcal{N}_{\rho_n}(b_n) \cup \mathcal{N}_{\rho_n}(c_n))$$

is connected. Choose $1 \le k \le 16$ such that

$$\max\{\operatorname{dist}(v_n^i, \partial \Delta_n') : i = 1, 2, 3\} \notin [kc, (k+5)c]$$

Apply kc iterations of the procedure $(*)$ from Step 1 to Δ_n', first for α_n', then for β_n' and finally for γ_n'. As α_n', β_n', γ_n' have pairwise disjoint kc-neighbourhoods, we obtain a connected complex $\Delta_n'' \subset \Delta_n'$ such that $\operatorname{hdist}_X(\Delta_n', \Delta_n'') = o(\lambda_n)$ and $\operatorname{dist}_{\Delta_n'}(\Delta_n'', \partial \Delta_n') = kc$.

Consider balls $\mathcal{N}_c(v_n^i) \subset \Delta_n''$, $i = 1, 2, 3$. If they are not disjoint, then either replace two of them with a larger ball $\mathcal{N}_{3c}(v_n^i)$ or replace all of them with a single ball $\mathcal{N}_{5c}(v_n^i)$ to obtain at most three disjoint balls, B_n^1, B_n^2 and B_n^3 of radii not greater than $5c$ and such that the complement

$$D_n = \Delta_n'' \setminus (B_n^1 \cup B_n^2 \cup B_n^3)$$

is disjoint from $\mathcal{N}_{c-1}(v_n^i)$, for $i = 1, 2, 3$. By the choice of ρ_n and k, these balls are disjoint from $\partial\Delta_n''$. Thus D_n is a disc with at most three holes, in particular it is connected.

Since the ball $\mathcal{N}_c(v) \subset \Delta_n$ is an equilaterally triangulated regular hexagon for every $v \in D_n$, by Theorem 3.5 S_n restricted to $\mathcal{N}_c(v)$ is an isometric embedding and by Lemma 5.6(1) $S_n(\mathcal{N}_c(v))$ is contained in some flat F_n^v. By the connectedness of D_n and Lemma 5.6(2) the flat F_n^v does not depend on v, thus the whole image of D_n is contained in some flat F_n. As $\Delta_n = \mathcal{N}_{o(\lambda_n)}(D_n)$, the image $\operatorname{Im} S_n$ is contained in $\mathcal{N}_{o(\lambda_n)}(F_n)$, in particular $\alpha_n \cup \beta_n \cup \gamma_n \subset \mathcal{N}_{o(\lambda_n)}(F_n)$. Thus $\alpha \cup \beta \cup \gamma = \lim_\omega(\alpha_n \cup \beta_n \cup \gamma_n) \subset \lim_\omega F_n$. □

Corollary 5.14. *Let G be a group acting properly discontinuously and cocompactly on a systolic complex X satisfying the Isolated Flats Property. Then G is relatively hyperbolic with respect to a collection of maximal virtually abelian rank 2 subgroups.*

Proof: In Theorem 5.13 we proved that X is asymptotically tree-graded with respect to the family $\mathcal{F}$ of thickenings of all flats in X (as the Hausdorff distance between a flat and its thickening is at most 1). By Fact 5.2 there are finitely many orbits of action of G on equivalence classes of flats in X. Choose flats $F_1, \ldots, F_m$ representing the orbits and an arbitrary vertex $x \in X$. Let $K \subset X^{(0)}$ be a (finite) set of representants of G acting on $X^{(0)}$ and define $q : X \to G$ such that $q(gK) = g$. The map q is a quasi-isometry and by Proposition 5.3 $\mathrm{Stab}(Th(F_i))$ acts cocompactly on $Th(F_i)$, so there is a constant c such that $\mathrm{hdist}_G(q(gF_i), g \cdot \mathrm{Stab}(Th(F_i))) \leq c$, for $g \in G$, $i = 1, \ldots, m$. By Theorem 5.1 in [DS] the group G is asymptotically tree-graded with respect to $q(F)$ for $F \in \mathcal{F}$, so it is asymptotically tree-graded with respect to $g \cdot \mathit{Stab}(Th(F_i))$, for $g \in G$, $i = 1, \ldots, m$. Corollary 5.4 and Theorem 5.11 complete the proof. □

5.3. Relative Fellow Traveller Property

Definition 5.15. (Relative Fellow Traveller Property) *Two paths $\gamma : [0, c] \to X$ and $\gamma' : [0, c'] \to X$ in a systolic complex X relatively δ-fellow travel if there are partitions:*

$$0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_n = c \;\; \text{and} \;\; 0 = t'_0 \leq t'_1 \leq t'_2 \leq \ldots \leq t'_n = c'$$

such that:

- *$\gamma(t_i)$ and $\gamma'(t'_i)$ are δ-close vertices for $0 \leq i \leq n$;*
- *subpaths $\gamma([t_i, t_{i+1}])$ and $\gamma'([t'_i, t'_{i+1}])$ are either Hausdorff δ-close or both are contained in $\mathcal{N}_\delta(F_i)$ for some flat F_i.*

A systolic complex X has the Relative Fellow Traveller Property *if for every $L, C > 0$ there is a constant $\delta = \delta(L, C)$ such that every pair of (L, C)-quasi geodesics with common endpoints δ-fellow travels.*

Corollary 5.16. *Every cocompact systolic complex X with the Isolated Flats Property, satisfies the Relative Fellow Traveller Property.*

Proof: By Theorem 5.13 X is asymptotically tree-graded with respect to a family of flats $\mathcal{F}$ consisting of one flat from every equivalence class. Let $L, C > 0$ and $\mathfrak{p} : [0, c] \to X$,

$\mathfrak{q} : [0, c'] \to X$ arbitrary (L, C)-quasi geodesics in X with common endpoints. By Lemma 4.25 in [DS] there exist constants $\tau > 1$ and $M > 0$, depending only on L and C, such that $\mathfrak{p} \subset \mathcal{N}_\tau(\mathrm{Sat}(\mathfrak{q}))$, where

$$\mathrm{Sat}(\mathfrak{q}) = \mathfrak{q} \cup \bigcup \{F \in \mathcal{F} : \ F \cap \mathcal{N}_M(\mathfrak{q}) \neq \emptyset\}$$

Thus there is a partition $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_n = c$, such that subpath $\mathfrak{p}[t_i, t_{i+1}]$ is contained either in $\mathcal{N}_\tau(F_{k_i})$ or in $\mathcal{N}_\tau(\mathfrak{q})$. This implies that $\mathfrak{p}(t_i) \in \mathcal{N}_\tau(\mathfrak{q})$, unless $\mathfrak{p}[t_{i-1}, t_i] \subset \mathcal{N}_\tau(F_{k_i})$ and $\mathfrak{p}[t_i, t_{i+1}] \subset \mathcal{N}_\tau(F_{k_{i+1}})$. However, in this case we apply Lemma 4.25 in [DS] to $\mathrm{Sat}(\mathfrak{p}(t_i))$, obtaining (by the Isolated Flats Property) that $\mathrm{dist}(\mathfrak{p}(t_i), \mathfrak{q}) < \psi(\tau M)$.

Hence there exist a partition $0 = t'_0 \leq t'_1 \leq t'_2 \leq \ldots \leq t'_n = c'$ and a constant $\delta = \delta(L, C)$ such that $d(\mathfrak{p}(t_i), \mathfrak{q}(t'_i)) < \delta$ and for $i = 0, \ldots, n-1$ either $\mathfrak{p}[t_i, t_{i+1}] \subset \mathcal{N}_\delta(F_i)$ or $\mathfrak{p}[t_i, t_{i+1}] \subset \mathcal{N}_\delta(\mathfrak{q})$. In the first case, by Lemma 4.15 in [DS], $\mathfrak{q}[t'_i, t'_{i+1}] \subset \mathcal{N}_{\tau\delta}(F_i)$. In the second case $\mathrm{hdist}(\mathfrak{p}[t_i, t_{i+1}], \mathfrak{q}[t'_i, t'_{i+1}]) < \delta'$ for some constant δ', depending only on L and C. $\square$

The following theorem summarizes the results of Section 5.

Theorem 5.17. *For a systolic complex X with a cocompact and properly discontinuous action of a group G the following are equivalent:*

(1) *X satisfies the Isolated Flats Property,*

(2) *X satisfies the Relative Fellow Traveller Property,*

(3) *X contains no isometrically embedded triplanes,*

(4) *G is relatively hyperbolic with respect to a family of maximal virtually abelian rank 2 subgroups.*

Proof: We proved (1) $\iff$ (3) (Theorem 5.7), (1) $\implies$ (2)+(4) (Corollaries 5.14 and 5.16). To show $\neg(3) \implies \neg(2)$ we need to construct for every $\delta > 0$ two (L, C)-quasi-geodesics in a triplane that do not δ-fellow travel. As any systolic triplane is quasi-isometric to the metric triplane (i.e. the geodesic space obtained by gluing three Euclidean half-planes by isometries on their boundaries) it suffices to do the construction for the metric triplane.

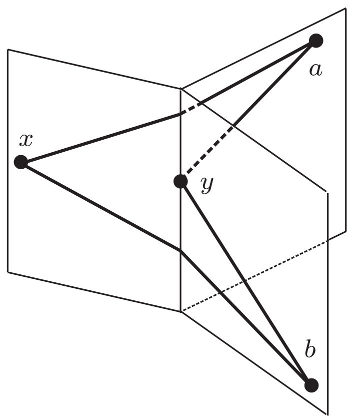

Figure 5.6.

An example presented in [Hr1] is the following: concatenations $xa * ay$ and $xb * by$ (in Figure 5.6) are two quasi-geodesics with endpoints x and y; by scaling the picture we obtain a sequence of (L, C)-quasi geodesics p_n and q_n which do not n-fellow travel.

As every maximal virtually abelian rank 2 subgroup of G is equal to $\mathrm{Stab}(Th(F))$ for some flat F (Corollary 3.4) and the stabilizer acts cocompactly on $Th(F)$ (Proposition 5.3), condition (4) implies that X is asymptotically tree-graded with respect to some collection of flats. Thus (4) $\Longrightarrow$ (3) is a consequence of Proposition 4.5 and the fact that an asymptotic cone of a triplane is homeomorphic to a triplane. $\square$

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