Isometries of systolic spaces

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Abstract: We prove that any isometry of a systolic complex either fixes a simplex (elliptic case) or has an invariant thick geodesic (hyperbolic case). This provides an alternative proof that finitely generated abelian subgroups of systolic groups are undistorted.

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Contents

1. Introduction

Systolic complexes were introduced by Tadeusz Januszkiewicz and Jacek Świątkowski in [JS1] and independently by Frédéric Haglund in [Ha]. They are connected simply connected simplicial complexes satisfying certain local combinatorial condition (see Definition 2.1 for details), which is a simplicial analogue of nonpositive curvature. Systolic complexes have many properties similar to properties of CAT(0)-spaces; however, systolicity neither implies, nor is implied by nonpositive curvature of the complex equipped with the piecewise euclidean metric for which simplices are regular euclidean simplices.

In this paper we study individual isometries of a systolic space. The first result of the paper is the following:

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Theorem 1.1. (see Theorem 3.5) *If g is a (simplicial) isometry of a systolic complex X then either g fixes a simplex or there is a g^n-invariant geodesic in X for some $n \geq 1$.*

If an isometry g fixes a simplex σ, then the barycentre of σ is its fix-point (the isometry is elliptic). Unfortunately, there are examples of non-elliptic isometries of systolic spaces which do not have a g-invariant geodesic (i.e. the power n in Theorem 1.1 is necessary).

Example 1.2. Let $k \geq 2$. Define a simplicial complex A_k such that such that $A_k^{(0)} = \mathbb{Z}$ and $\sigma \subset A_k^{(0)}$ spans a simplex if and only if $|a - a'| \leq k$ for all $a, a' \in \sigma$. The complex A_k is systolic (see Definition 2.1) and the isometry $g : A_k \to A_k$ induced by the map $\mathbb{Z} \ni x \mapsto x + 1 \in \mathbb{Z}$ has no invariant geodesics.

The above example motivates us to introduce the concept of thick axis (which is a subcomplex at Hausdorff distance at most 1 from a geodesic) and state the elliptic/hyperbolic dichotomy for systolic spaces as follows:

Theorem 1.3. (see Theorem 3.9) *If g is a (simplicial) isometry of a systolic complex X then either g fixes a simplex (elliptic case) or there is a g-invariant 'thick axis', i.e. a full subcomplex $A \subset X$ isomorphic to A_k (defined in Example 1.1) for some $k \geq 1$ (hyperbolic case).*

As a corollary we obtain a proof (alternative to [JS1]) of the fact that infinite cyclic subgroups of a group acting cocompactly and properly discontinuously on a systolic complex are undistorted (see Corollary 3.10). The rest of the paper is devoted to the proof of Theorem 1.3.

2. Systolic complexes

Let X be a simplicial complex and σ a simplex of X. The *link* of X at σ, denoted X_σ, is a subcomplex of X consisting of all simplices that are disjoint from σ and together with σ span a simplex of X.

A simplicial complex X is *flag* if every finite set of its vertices pairwise connected by edges spans a simplex of X. A subcomplex $Y \subset X$ is *full* if any simplex $\sigma \subset X$ with all vertices in Y is contained in Y.

A *cycle* in X is a subcomplex γ isomorphic to a triangulation of a circle. The *length* of γ (denoted $|\gamma|$) is the number of its edges. A *diagonal* of a cycle is an edge joining its two nonconsecutive vertices.

Whenever we refer to a metric on a simplicial complex, we actually mean the 1-skeleton of the complex equipped with the combinatorial metric (i.e. the geodesic metric in which all edges have length 1). Thus for a simplicial complex X the symbol 'd_X' denotes the combinatorial metric on $X^{(1)}$. Moreover, referring to a geodesic in a simplicial complex X, we mean a geodesic in $X^{(1)}$ having both endpoints in $X^{(0)}$.

Definition 2.1. (see [JS2]) *A simplicial complex X is called:*

- 6-large *if it is flag and every cycle* γ *in* X *of length* $4 \leq |\gamma| < 6$ *has a diagonal;*
- locally 6-large *if the link at every (nonempty) simplex of* X *is 6-large;*
- systolic *if it is locally 6-large, connected and simply connected.*

An equivalent definition of systolicity can be obtained by replacing words 'locally 6-large' with '6-large':

Fact 2.2. ([JS1], Proposition 1.4) *Every systolic complex is 6-large. In particular, a cycle of length smaller than 6 in a systolic complex bounds a triangulated disc with no internal vertices.*

2.1. Minimal surfaces

The main tool used in the paper is the theory of minimal surfaces in systolic complexes, developed in [E]. A minimal surface is a simplicial map $S : \Delta \to X$ from a triangulation Δ of a 2-disc to a systolic complex X with the property that Δ has the minimal number of triangles among all such maps extending $S|_{\partial\Delta}$. By Lemma 4.2 in [E] any simplicial map $f : S^1 \to X$ from a triangulated circle can be extended to a minimal surface (the extension need not be unique) and the domain Δ of a minimal surface is a systolic complex.

For any triangulation Δ of a disc we define the *defect at* v by the following formula:

$$\mathrm{def}\,(v) = \begin{cases} 6 - \#\{\text{triangles in } \Delta \text{ containing } v\}, & \text{if } v \notin \partial\Delta \\ 3 - \#\{\text{triangles in } \Delta \text{ containing } v\}, & \text{if } v \in \partial\Delta \end{cases}$$

We call vertices *(non)positive*, *(non)negative* or *zero* if their defects are such. The term 'the sum of the defects of Δ along a boundary line $l \subset \partial\Delta$' will be used to denote the sum of the defects at all vertices of l but the endpoints.

Lemma 2.3. (Gauss-Bonnet Lemma) *If* Δ *is a triangulation of a 2-disc, then:*

$$\sum_{v \in \Delta^{(0)}} \mathrm{def}(v) = 6.$$

In particular, if Δ *is a systolic triangulation, then the sum of defects at boundary vertices is at least 6, with the equality if and only if* Δ *has no internal vertices of negative defect.*

The proof of Theorem 1.2 is based on the following two theorems, which summarizes the relevant results from sections 3, 4 and 5 of [E]:

Theorem 2.4. *Let* X *be a systolic complex and* P *be a triangulation of a strip* $\mathbb{R} \times I$ *such that:*

(i) *every vertex* $v \in \partial P$ *has defect* -1, 0 *or* 1*;*
(ii) *every internal vertex* $v \in P$ *has defect 0;*
(iii) *in each boundary component of* P*, between any two nonzero vertices of the same sign there is a vertex of the opposite sign;*

(iv) ∂P *is a full subcomplex of* P.

Then any simplicial map $f : P \to X$ *with the following property:*

($*$) *for every internal vertex* $v \in P$ *and for every edge* $uw \subset P$ *with both endpoints at internal vertices* $f|_{N(v)}$ *and* $f|_{N(uw)}$ *are minimal surfaces,*

maps each connected component of ∂P *to a geodesic in* X.

Here and subsequently $N(K)$ denotes the subcomplex of X being the union of all closed simplices intersecting K.

Proof: The complex P can be presented as an increasing union of simplicial discs P_n such that any internal vertex $v \in P_n$ has defect 0, any boundary vertex $v \in P_n$ has defect at least -1 and any two negative vertices on ∂P_n are separated by a positive one. By Lemma 3.5 in [E] simplicial maps $f|_{P_n} : P_n \to X$ satisfy the assumption of Theorem 4.12 in [E]. By Proposition 4.7(2), Theorem 4.12 and Corollary 4.11(2) in [E] the intersection of P_n with a boundary component of P is mapped by f to a geodesic in X. Since $P_n \subset P_{n+1}$ for $n = 1, 2, \ldots$ and $P = \bigcup_n P_n$, each boundary component of P is mapped to a geodesic in X. □

Theorem 2.5. (see Theorem 5.2 in [E]) *Let* $\mathbb{E}^2_\triangle$ *be the triangulation of the euclidean plane by congruent equilateral triangles,* X *a systolic complex and* $F : \mathbb{E}^2_\triangle \to X$ *a simplicial map. If* $F|_{N(v)}$ *is an isometric embedding for any vertex* $v \in \mathbb{E}^2_\triangle$*, then* F *is an isometric embedding.*

3. Classification of isometries

Let X be a systolic complex and g a (simplicial) isometry of X. Let us address first the elliptic case.

Proposition 3.1. *If* g *is a (simplicial) isometry of a systolic complex* X *and* g^n *fixes a simplex of* X *for some* $n \geq 1$*, then* g *fixes a simplex of* X.

Proof: Choose a vertex $v \in X$ and an integer $R > 0$ sufficiently large so that

$$Y = \mathcal{N}_R(v) \cap \mathcal{N}_R(g(v)) \cap \ldots \cap \mathcal{N}_R(g^{n-1}(v)) \neq \emptyset$$

The subcomplex $Y \subset X$ is g-invariant. By Corollary 4.10 in [HS] balls around vertices in systolic complexes are geodesically convex, so their intersection $Y \subset X$ is also such. Hence by Lemma 7.2 in [JS1] the complex Y is systolic (by Proposition 4.9 in [HS] notions of convexity and geodesic convexity coincide). Thus $H_i(Y) = 0$, for $i = 1, 2, \ldots$ (by Theorem 4.1 in [JS1] systolic complexes are contractible) and by the Lefschetz Fix-Point Theorem the isometry $g|_Y : Y \to Y$ has a fix-point $y \in Y$. Since g preserves a simplicial structure it fixes the minimal simplex containing y. □

Thus, in the subsequent part of the paper, we only need to consider free actions of $\mathbb{Z}$ on systolic complexes.

3.1. The minimal displacement

For a (simplicial) isometry g of a simplicial complex X we define the *minimal displacement*:

$$|g| = \min_{x \in X^{(0)}} d_X(x, g(x))$$

The full subcomplex of X spanned by all vertices $v \in X$ satisfying $d_X(v, g(v)) = |g|$ we denote by $\mathrm{Min}(g)$. It is clearly non-empty. Below (Propositions 3.3 and 3.4) we prove that $\mathrm{Min}(g)$ is a systolic complex and its 1-skeleton is isometrically embedded in X.

Fact 3.2. *Let g be a simplicial isometry of a systolic complex X without fix-points. Choose a vertex $v \in \mathrm{Min}(g)$, a geodesic $\alpha \subset X^{(1)}$ joining v with $g(v)$ and consider a simplicial path $\gamma : \mathbb{R} \to X$ (where $\mathbb{R}$ is given a simplicial structure with $\mathbb{Z}$ as the set of vertices) being the concatenation of geodesics $g^n(\alpha)$, $n \in \mathbb{Z}$. Then γ is a $|g|$-geodesic (i.e. $d(\gamma(a), \gamma(b)) = |a-b|$ if a, b are such integers that $|a - b| \leq |g|$). In particular, $\mathrm{Im}(\gamma) \subset \mathrm{Min}(g)$.*

Proof: We prove the statement for $|a-b| = |g|$ (this implies the general case). Then, by the construction of γ, either $\gamma(b) = g(\gamma(a))$ or $\gamma(a) = g(\gamma(b))$, thus we have $d(\gamma(a), \gamma(b)) \geq |g|$. The opposite inequality follows from the fact that γ is a simplicial map. $\square$

Proposition 3.3. *For a (simplicial) isometry g of a systolic complex X having no fix-points the 1-skeleton of $\mathrm{Min}(g)$ is isometrically embedded into X.*

Proof: Suppose the 1-skeleton of $\mathrm{Min}(g)$ is not isometrically embedded. Then there exist vertices $v, w \in \mathrm{Min}(g)$ such that no geodesic in X with endpoints v and w is contained in $\mathrm{Min}(g)$. Choose v and w so that $d_X(v, w)$ is minimal (clearly $d_X(v, w) > 1$). Join v with $g(v)$, w with $g(w)$ and v with w by geodesics α, β and ξ, respectively. Then $g(v)$ is joined with $g(w)$ by $g(\xi)$. Denote by v' and w' the vertices on ξ connected by edges with v and w, respectively, as in Figure 3.1 (possibly $v' = w'$).

By minimality of $d(v, w)$ geodesics α and ξ intersect only at the endpoints, since $\alpha \subset \mathrm{Min}(g)$ by Fact 3.2. The same holds for geodesics α and $g(\xi)$, β and ξ, β and $g(\xi)$.

Suppose there is a vertex $x \in \xi \cap g(\xi)$. Then $g(x) \in g(\xi)$ and $g(x) \neq x$, since g has no fix-points. We may assume, not losing generality, that $g(v)$, x, $g(x)$ and $g(w)$ lie on $g(\xi)$ in this order. Then

$$d(x, g(x)) = d(g(v), g(x)) - d(g(v), x) = d(v, x) - d(g(v), x) \leq d(v, g(v)) = |g|$$

so $x \in \mathrm{Min}(g)$, contradicting the minimality of $d(v, w)$.

Thus geodesics α, β, ξ, $g(\xi)$ either are pairwise disjoint but the endpoints, or α and β have nonempty intersection. Consider the case when α and β can be chosen so that $\alpha \cap \beta \neq \emptyset$ and let the intersection be maximal (Figure 3.1(a)). The intersection is a geodesic with endpoints p and q. Decompose $\alpha = \alpha' \cup [p, q] \cup \alpha''$ and $\beta = \beta' \cup [p, q] \cup \beta''$. Let $S' : \Delta' \to X$ be a minimal surface spanning the cycle $\alpha' * \beta' * \xi$ and $S'' : \Delta'' \to X$ a minimal surface spanning the cycle $\alpha'' * \beta'' * g(\xi)$.

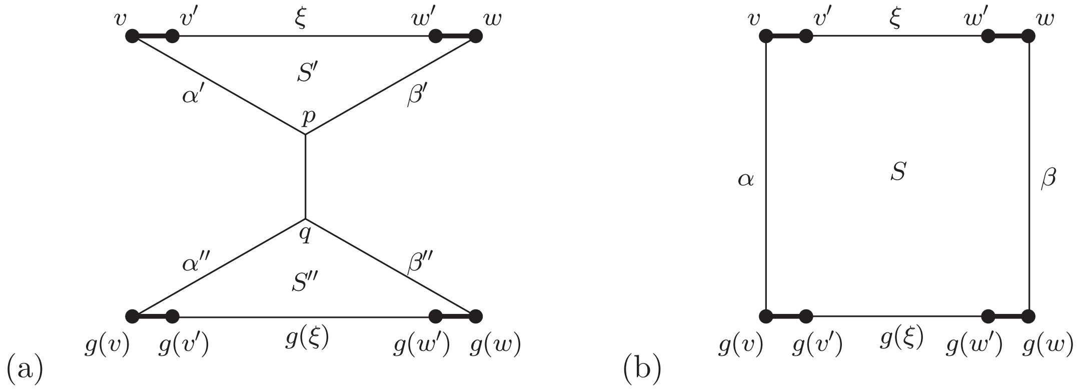

Figure 3.1.

Choose α, β, ξ minimizing the area of Δ' and in the case of equal areas – minimizing the area of Δ''. Thus boundary vertices of Δ' different from v, w, p have nonpositive defects, so by the Gauss-Bonnet Lemma $\mathrm{def}_{\Delta'}(v) = \mathrm{def}_{\Delta'}(w) = \mathrm{def}_{\Delta'}(p) = 2$ and defects at all other vertices of Δ' are equal to 0. The similar calculation proves that either $\mathrm{def}_{\Delta''}(g(v)) = 2$ or $\mathrm{def}_{\Delta''}(g(w)) = 2$ (defects at vertices of α'' and β'' different from the endpoints are nonpositive and the sum of defects along $g(\xi)$ does not exceed 1, as $g(\xi)$ is a geodesic – see Remark 3.1 in [E]). Thus either v' with $g(v')$ or w' with $g(w')$ can be connected by a polygonal path of length $d(v, g(v)) = d(w, g(w)) = |g|$, contradicting the assumption $v', w' \notin \mathrm{Min}(g)$.

Now assume that we cannot chose α and β with non-empty intersection. Denote by $S : \Delta \to X$ a minimal surface spanning the cycle $\alpha * \xi^{-1} * \beta * g(\xi)$ and choose α, β and ξ minimizing the area of Δ (Figure 3.1(b)). Thus the defects of vertices of α and β different from their endpoints are nonpositive. The sum of defects along the polygonal line $\bar{\alpha}$ composed of edges vv' and $g(v)g(v')$ and a geodesic α cannot exceed 2. Otherwise either $\mathrm{def}(v) = \mathrm{def}(g(v)) = 2$ and $d(v', g(v')) \leq d(v, g(v)) = |g|$ or $\mathrm{def}(v) = 2$, $\mathrm{def}(g(v)) = 1$ (or conversely) and the defect at any vertex of α different from its endpoints is 0 – then also $d(v', g(v')) \leq d(v, g(v)) = |g|$. In both cases $v' \in \mathrm{Min}(g)$, contradicting the minimality of $d(v, w)$. Similarly the sum of defects along the path $\bar{\beta}$ composed of β and edges ww' and $g(w)g(w')$ does not exceed 2.

Choose a vertex $x \in \xi$ (different from the endpoints). If $\mathrm{def}(x) + \mathrm{def}(g(x)) > 0$, then at least one of the defects (say $\mathrm{def}(x)$) is equal to 1 (by geodesity of ξ and $g(\xi)$ we have $\mathrm{def}(x) \leq 1$ and $\mathrm{def}(g(x)) \leq 1$). Modifying boundary geodesics ξ and $g(\xi)$ by cutting off Δ two triangles adjacent to x and gluing their g-images to $g(x)$ we obtain a surface $S' : \Delta' \to X$, where Δ' has the same area as Δ, but S' is not minimal ($x \notin \mathrm{Min}(g)$ implies $d(x, g(x)) > 1$, so $g(x) \in \Delta'$ is an internal vertex adjacent to less than 6 triangles). This contradicts the minimality of the area of Δ and therefore $\mathrm{def}(x) + \mathrm{def}(g(x)) \leq 0$ for any $x \in \xi$ different from the endpoints. This with the calculation from the previous paragraph shows that the sum of defects at vertices of $\partial\Delta$ is not greater than 4, while by the Gauss-Bonnet Lemma it is at least 6. Hence any two vertices $v, w \in \mathrm{Min}(g)$ can be connected by a geodesic in X contained in $\mathrm{Min}(g)$. $\square$

Proposition 3.4. *If $Y \subset X$ is a full subcomplex of a systolic complex X such that $Y^{(1)}$*

is isometrically embedded into X, then Y is a systolic complex. In particular, for any simplicial isometry g of X having no fix-points the subcomplex $\mathrm{Min}(g)$ *is systolic.*

Proof: The complex Y is 6-large (as a full subcomplex of a systolic complex) and connected (since $Y^{(1)} \subset X$ is an isometric embedding). Thus we need only to prove that it is simply connected.

Let γ be the shortest loop in $Y^{(1)}$ that is not contractible in Y. Then γ is embedded and any subpath of γ of length not greater than $\frac{1}{2}|\gamma|$ is a geodesic (as otherwise there are vertices $v, w \in \gamma$ disconnecting γ into subpaths γ_1 and γ_2 such that for a geodesic ξ connecting v and w loops $\gamma_1 \cup \xi$ and $\gamma_2 \cup \xi$ are homotopically trivial, by minimality of $|\gamma|$). Thus the loop γ can be covered by not more than 5 subpaths $\gamma_1, \ldots, \gamma_5$ which are geodesics, in such a way that every vertex of γ occurs as an internal vertex of γ_i for exactly one i. It follows that for a minimal surface $S : \Delta \to X$ spanning γ, the sum of defects at vertices of $\partial\Delta$ is at most 5 (since the sum of defects along any geodesic in $\partial\Delta$ is at most 1 – see Remark 3.1 in [E]), contradicting the Gauss-Bonnet Lemma. Therefore, there are no homotopically non-trivial loops in Y. The last part of the proposition follows from Proposition 3.3. □

3.2. An invariant geodesic

Theorem 3.5. *Let g be a non-elliptic simplicial isometry (i.e. there are no simplices fixed by g) of a uniformly locally finite systolic complex X . Then there is a g^n-invariant geodesic, for some $n \geq 1$.*

Proof: By Propositions 3.3 and 3.4 we may assume, without losing generality, that $\mathrm{Min}(g) = X$. Denote by G the cyclic group of isometries generated by g. Since g^n is non-elliptic for any $n \geq 1$ (Proposition 3.1), the action of G on X is free.

Case 1: *The action of G on $X = \mathrm{Min}(g)$ is not cocompact and $|g| > 3$.*

Choose vertices $v_1, v_2 \in X$ such that $(|g|+1)$-neighbourhoods of the orbits Gv_1 and Gv_2 are disjoint. Connect v_1 with $g(v_1)$ and v_2 with $g(v_2)$ by geodesics α_1 and α_2, respectively. Let β be a geodesic connecting v_1 and v_2. By contractibility of X and the relative Simplicial Approximation Theorem there exists a map $p : \Delta \to X$, where Δ is a triangulation of a disc and p maps $\partial\Delta$ onto the closed path being the concatenation $\alpha_1 * \beta^{-1} * \alpha_2 * g(\beta)$. Since $|g| > 3$, the quotient space X/G is a simplicial locally 6-large complex and we obtain as a quotient of p a simplicial map $f : A \to X/G$, where A is a triangulation of an annulus. Now we modify A and f applying four types of operations:

(a) If there exists in A a cycle ξ of length 3 not bounding a triangle in A, then by the assumption $|g| > 3$ and flagness of X/G the cycle ξ is homotopically trivial, so it disconnects A into two components, one of them being a triangulation of a disc. By replacing this component with a single triangle we obtain another triangulation of an annulus.

(b) If any cycle of length 3 in A bounds a triangle and there is an internal vertex $v \in A$ adjacent to 4 or 5 triangles, we cut out the open star of v and glue the filling without internal vertices, instead (this is possible, since X/G is locally 6-large), obtaining another simplicial triangulation of an annulus.

(c) If there exists an internal vertex $v \in A$ with the property that $f(\partial N(v))$ can be filled without internal vertices, then we apply the procedure from (b) to the star of v.

(d) If there exist two internal vertices $v, w \in A$ connected by an edge such that $f(\partial N(vw))$ has a filling with at most 1 internal vertex, then we cut out the interior of $N(vw)$ and glue in such a filling.

As we modify A, we modify f. Since each operation lowers the number of vertices in A, the procedure terminates. Thus we obtain a map $f' : A' \to X/G$, where A' is a triangulation of an annulus in which every internal vertex is adjacent to at least 6 triangles and f' satisfies the condition $(*)$ from Theorem 2.4. The boundary $\partial A'$ is the disjoint union of two circles: c_1 and c_2, each of length $|g|$. Let $\widetilde{f'} : \widetilde{A'} \to X$ be the universal covering of f', where $\widetilde{A'}$ is a triangulation of a strip ($\widetilde{X/G} = X$, since by Theorem 4.1 in [JS1] systolic complexes are contractible). By the choice of α_1 and α_2 and by Fact 3.2 paths $\widetilde{f'}(\widetilde{c_1})$ and $\widetilde{f'}(\widetilde{c_2})$ are $|g|$-geodesics in X, so there are no vertices of defect 2 on $\partial A'$ and every arc in $\partial A'$ with both endpoints at vertices of defects $+1$ contains a vertex of negative defect. Thus the sum of the defects at boundary vertices of A' is nonpositive and by the construction every internal vertex has a nonpositive defect. Since the Euler characteristic of an annulus is 0, the combinatorial Gauss-Bonnet Theorem implies that every internal vertex of A' has defect 0 and the sum of defects at boundary vertices is equal to 0. Hence there are no vertices of defect less than -1 on $\partial A'$ and every two vertices of defect -1 on $\partial A'$ are separated by a vertex of defect $+1$. Thus the strip $\widetilde{A'}$ satisfies the assumptions of Theorem 2.4 (condition (iv) is fulfilled, since by the choice of v_1 and v_2 we have $\mathrm{dist}_X(\widetilde{f'}(\widetilde{c}_1), \widetilde{f'}(\widetilde{c}_2)) > 3$). Hence each boundary component of the strip is mapped to a geodesic in X and by the construction it is a g-invariant geodesic.

Case 2: *The action of G on $X = \mathrm{Min}(g)$ is not cocompact and $|g| \leq 3$.*

Consider a sequence of subcomplexes

$$X_0 = X, \quad X_n = \mathrm{Min}_{X_{n-1}}(g^n), \text{ for } n = 1, 2, \ldots$$

By Propositions 3.3 and 3.4 the subcomplex $X_n \subset X_{n-1}$ is a systolic g-invariant isometrically embedded subcomplex. Notice, that the minimal displacement of g^n in X_n satisfies

$$|g^n|_{X_n} = |g^n|_{X_k} \text{ for any } k > n$$

As G acts freely on X_n for every n and X_n is uniformly locally finite, there exists n, such that $|g^n|_{X_n} > 3$. Applying Case 1 we obtain a g^n-invariant geodesic l in X_n. As $X_n \subset X$ is a g-invariant and isometrically embedded subcomplex, l is a g^n-invariant geodesic in X.

Case 3: *The action of G on $X = \mathrm{Min}(g)$ is cocompact.*

Since any ball in X contains a finite number of vertices (by local finiteness of X) and $\mathrm{diam}(X) = \infty$ (the action of $G \cong \mathbb{Z}$ is free), there exists (by the standard diagonal argument) a bi-infinite geodesic l in $X^{(1)}$.

As $G \cong \mathbb{Z}$ acts freely cocompactly on X, the space X has 2 ends. Thus there exists a finite subcomplex $B \subset X$ disconnecting X such that any bi-infinite geodesic in X intersects B. Let n be the number of vertices in B and denote $B_i := g^i(B)$, for $i = 1, 2, \ldots$. Then for any i there are two geodesics among $l, g(l), \ldots, g^n(l)$ with a common vertex in B_i. Hence there are geodesics $g^j(l)$ and $g^{j+k}(l)$ with an infinite intersection. The same property hold for l and $g^k(l)$. The existence of g^k-invariant geodesic follows from the subsequent lemma. □

Lemma 3.6. *Let f be a simplicial isometry of a locally finite simplicial complex X and γ a geodesic in $X^{(1)}$. If $\gamma \cap f(\gamma)$ contains infinitely many vertices, then there is an f-invariant geodesic in X.*

Proof: We construct recursively a sequence of geodesics γ_i, $i = 0, 1, 2, \ldots$ such that $\gamma_i \cap f(\gamma_i)$ has infinitely many vertices and contains a geodesic of length at least i. Put $\gamma_0 := \gamma$. Suppose we have already constructed γ_i and $[a_i, b_i] \subset \gamma_i \cap f(\gamma_i)$ is a maximal geodesic in the intersection. We may assume, not losing generality, that b_i separates $f(b_i)$ and a_i on $f(\gamma_i)$. Then a_i separates b_i and $f^{-1}(a_i)$ on γ_i. Since γ_i and $f(\gamma_i)$ have infinite intersection, there is $x_i \in \gamma_i \cap f(\gamma_i)$ such that either $f(b_i)$ separates b_i and x_i on $f(\gamma_i)$ or $f^{-1}(a_i)$ separates a_i and x_i on γ_i. In the first case we obtain γ_{i+1} from γ_i by replacing the segment with endpoints b_i and x_i with the segment from $f(\gamma_i)$ with the same endpoints. Then there is a common segment $[a_i, f(b_i)] \subset \gamma_{i+1} \cap f(\gamma_{i+1})$. In the second case we define $f(\gamma_{i+1})$ to be $f(\gamma_i)$ with the segment with endpoints x_i and a_i replaced with the segment of γ_i having the same endpoints. Then there is a common segment $[f^{-1}(a_i), b_i] \subset \gamma_{i+1} \cap f(\gamma_{i+1})$.

Now fix an arbitrary vertex $v \in X$. By local finiteness of X and the standard diagonal argument we can choose a subsequence of geodesics γ'_i, such that the sequences γ'_i and $f(\gamma'_i)$ are convergent (uniformly on compact sets) and $\gamma'_i \cap \mathcal{N}_i(v) = f(\gamma'_i) \cap \mathcal{N}_i(v)$ for $i = 1, 2, \ldots$. Hence both sequences converges to the same geodesic $\bar{\gamma}$, which therefore satisfies $f(\bar{\gamma}) = \bar{\gamma}$. □

Remark 3.7. *If there exists a g^n-invariant geodesic in X, then for any vertex $x \in \mathrm{Min}(g^n) \subset X$ there exists a g^n-invariant geodesic passing through x.*

Proof: We construct a polygonal g^n-invariant line l passing through x as in Fact 3.2. By the existence of a g^n-invariant geodesic and by the minimal displacement of x the triangle inequality implies that l is also a geodesic. (see Step 3 in the proof of Theorem 6.1 in [E] for details). □

3.3. Thick axis

Fix some integer $k \geq 2$. Recall that A_k is a simplicial complex with $A_k^{(0)} = \mathbb{Z}$ such that $\sigma \subset \mathbb{Z}$ spans a simplex if and only if $|a - a'| \leq k$ for all $a, a' \in \sigma$. A *thick geodesic* in a systolic complex X will be the full subcomplex $A_k \subset X$, for some $k \geq 1$ such that

$$a - a' = jk,\ j \in \mathbb{Z}\ \Rightarrow\ d_X(a, a') = d_{A_k}(a, a')$$

Fact 3.8. *A thick geodesic $A_k \subset X$ is at Hausdorff distance 1 from an ordinary geodesic in X. Any isometry of A_k which is not elliptic is of the form $x \mapsto x + n$ for some $n \in \mathbb{Z}$.*

Proof: The first follows from the fact that elements $jk \in A_k$, $j \in \mathbb{Z}$ span a geodesic in A_k, hence in X. The second is implied by the fact that for $a, a' \in \mathbb{Z} = A_k^{(0)}$ the number $|a - a'|$ can be described as

$$|a - a'| = \operatorname{diam}\Big(\bigcap_{a,a' \in \sigma,\ \dim \sigma = k} \sigma \Big)$$

where σ is a simplex in A_k. $\square$

The subsequent theorem is the main result of the paper.

Theorem 3.9. *Any simplicial isometry g of a systolic complex X either fixes a simplex (elliptic case) or fixes a thick geodesic (hyperbolic case).*

Proof: By Theorem 3.5 if g is non-elliptic, then there is a g^n-invariant geodesic in X, for some $n \geq 1$. Let n be minimal. If $n = 1$ then g is fixes ordinary geodesic (which is isomorphic to A_1), thus suppose $n > 1$.

Denote by $\mathbb{E}^2_\triangle$ the triangulation of the euclidean plane by congruent equilateral triangles.

Step 1: *There exist:*

(i) *a group $T \cong \mathbb{Z}^2$ of isometries of $\mathbb{E}^2_\triangle$, generated by translations τ and σ,*

(ii) *a simplicial map $F : \mathbb{E}^2_\triangle \to X$ such that $F \circ \tau = g^n \circ F$ and $F \circ \sigma = g \circ F$,*

(iii) *a τ-invariant geodesic $m \subset \mathbb{E}^2_\triangle$ such that $F(m) \subset X$ is a g^n-invariant geodesic.*

Let l be a g^n-invariant geodesic in X and denote $l' = g(l)$. Choose vertices $x \in l$, $x' \in l'$ and join them by a geodesic γ in X. Denote by α and α' subgeodesics of l and l', connecting x with $g^n(x)$ and x' with $g^n(x')$, respectively. Let $f : \Delta \to X$ be a simplicial map, where Δ is a simplicial disc, such that f maps $\partial\Delta$ onto the closed path $\alpha * \gamma^{-1} * \alpha' * g^n(\gamma)$ and the area of Δ is minimal (such a map exists, since X is contractible as a systolic complex). Let x, x' and γ be chosen so that the area of Δ is minimal.

Gluing maps $(g^n)^i \circ f$, for $i \in \mathbb{Z}$ we obtain a g^n-equivariant simplicial map $f' : S \to X$, where S is a triangulation of a strip $\mathbb{R} \times I$ with g^n acting on it by translation and such that each boundary component of S is mapped to a geodesic in X. By systolicity of X every internal vertex of S is adjacent to at least 6 triangles (internal vertices of $\Delta \subset S$ are such, since the area of Δ is minimal, vertices on γ are such since γ was chosen to minimize the area of Δ).

Choose l to be a g^n-invariant geodesic minimizing the area of Δ. Denote boundary components of S by m and m' (where $f'(m) = l$ and $f'(m') = l'$) and define the isomorphism $p : m \to m'$ satisfying $f' \circ p = g \circ f'$.

We prove that $\operatorname{def}(x) + \operatorname{def}(p(x)) \leq 0$ for all vertices $x \in m$. This is immediate in the case when $|g^n| = 1$, as then by g^n-invariance and geodesity of m and m', any boundary vertex $x \in \partial S$ has a nonpositive defect. Thus assume $|g^n| > 1$. Since m and m' are geodesics in S, there is no vertices on ∂S of defect 2. Consider a vertex $x \in m \subset S$ of defect 1. If the defect at $p(x) \in m' \subset S$ is nonnegative, then we can modify S by cutting out triangles adjacent to vertices $(g^n)^i(x) \in m \subset S$ (open stars of these vertices are disjoint, as $|g^n| > 1$), for $i \in \mathbb{Z}$ and gluing their images at vertices $p((g^n)^i(x)) \in m' \subset S$, such that the modified map $\bar{f} : \bar{S} \to X$ is g^n-equivariant and maps ∂S to the disjoint union of two geodesics $\bar{l}$ and $g(\bar{l})$. If $|g| > 1$ then $\bar{S}$ contains a vertex adjacent to less than 6 triangles, which contradicts the minimality of the area of Δ (by systolicity of X). If $|g| = 1$, then l and $g(l)$ are Hausdorff 1-close and disjoint, what implies $\operatorname{def}(x) + \operatorname{def}(p(x)) \leq 0$ for all vertices $x \in m$, completing the proof of these inequality.

Now by the Gauss-Bonnet Lemma applied to subcomplexes of S bounded by two distant geodesics joining m and m' we obtain $\operatorname{def}(x) + \operatorname{def}(p(x)) = 0$ for all vertices $x \in m$ and any internal vertex $v \in S$ is adjacent to exactly 6 triangles. Therefore gluing maps $g^i \circ f'$, for $i \in \mathbb{Z}$, we obtain a simplicial map $F : \mathbb{E}^2_\triangle \to X$ satisfying (i)–(iii).

Step 2: *Choose T, F and m satisfying conditions (i)–(iii) from Step 1 so that the number of orbits of the action of T on the 0-skeleton of $\mathbb{E}^2_\triangle$ is minimal. Then m and $\sigma(m)$ are Hausdorff 1-close.*

Since the τ-invariant geodesic $m \subset \mathbb{E}^2_\triangle$ is mapped by F to a g^n-invariant geodesic in X, any τ-invariant geodesic $\bar{m} \subset \mathbb{E}^2_\triangle$ is mapped to a g^n-invariant geodesic in X. This follows from the following inequality (satisfied for all $x \in m$, $\bar{x} \in \bar{m}$):

$$N \cdot d(F(\bar{x}), F(\tau^i(\bar{x}))) \geq N \cdot d(F(x), F(\tau^i(x))) - 2d(F(x), F(\bar{x})), \text{ for any } N \in \mathbb{Z}$$

If $F|_{N(v)}$ was an isometric embedding for every vertex $v \in \mathbb{E}^2_\triangle$, then by Theorem 2.5 the map F would be an isometric embedding. However, F is not even injective, hence there is a vertex $v \in \mathbb{E}^2_\triangle$ such that the map $F|_{N(v)}$ is not an isometric embedding (so by systolicity of X the closed path $F(\partial N(v))$ can be filled by without internal vertices). If the distance between m and $\sigma(m)$ is greater than 1, then there is a τ-invariant geodesic $\bar{m}$ disjoint from the orbit Tv. Thus repeating the procedure from Step 1 starting with $\bar{l} = F(\bar{m})$ instead of $l = F(m)$ we obtain a contradiction with the minimality of the number of T-orbits. Therefore m and $\sigma(m)$ are Hausdorff 1-close.

Step 3: *If T, F and m are chosen as in Step 2, then $m \subset \mathbb{E}^2_\triangle$ is a convex geodesic.*

We introduce a coordinate system on $\mathbb{E}^2_\triangle$ by choosing a vertex $0 \in m \subset \mathbb{E}^2_\triangle$ and two edges of $\mathbb{E}^2_\triangle$ with endpoint 0 forming a euclidean angle of measure $\frac{\pi}{3}$ – they represent base vectors e_1 and e_2. Let the coordinates of the vectors $\overrightarrow{0\tau(0)}$ and $\overrightarrow{0\sigma(0)}$ be (α, β) and (a, b), respectively, $\alpha, \beta, a, b \in \mathbb{Z}$. We may choose e_1 and e_2 so that $\alpha, \beta \geq 0$.

We have to prove that either $\alpha = 0$ or $\beta = 0$. Assume, on the contrary, that $\alpha, \beta > 0$. Then for any vertices $p_0, p_1 \in \mathbb{E}^2_\triangle$ such that $p_i = (x_i, y_i)$, $i = 0, 1$ we have

$$x_0 \leq x_1 \wedge y_0 \leq y_1 \implies d_{\mathbb{E}^2_\triangle}(p_0, p_1) = d_X(F(p_0), F(p_1)) \tag{3.1}$$

since there exist integers $j < k$ such that p_0 and p_1 lie in this order on a geodesic joining $\tau^j(0)$ and $\tau^k(0)$, and the latter vertices lie on a geodesic m which is mapped to a geodesic in X.

If $a, b \geq 0$ or $a, b \leq 0$, then a geodesic in $\mathbb{E}^2_\triangle$ passing through $\sigma^k(0) = (ka, kb)$ for $k \in \mathbb{Z}$ is mapped by F to a geodesic γ in X (by (3.1)) and $\gamma \cap g(\gamma)$ has infinitely many vertices (since $F \circ \sigma = g \circ F$). By Lemma 3.6 there is a g-invariant geodesic in X, contrary to the assumption.

The only case left is when a and b are of different signs. Without loss of generality we can assume $a < 0 < b$. By (3.1) the set of vertices of any geodesic joining 0 and $\tau(0) = (\alpha, \beta)$ is contained in

$$\{xe_1 + ye_2 : 0 \leq x \leq \alpha, 0 \leq y \leq \beta,\ x, y \in \mathbb{Z}\}.$$

Thus the geodesic m passing through $\tau^k(0) = (k\alpha, k\beta)$ for $k \in \mathbb{Z}$ is contained in

$$P = \{xe_1 + ye_2 :\ k\alpha \leq x \leq (k+1)\alpha,\ k\beta \leq y \leq (k+1)\beta,\ x, y, k \in \mathbb{Z}\}.$$

Since $\sigma(m)$ is at Hausdorff distance 1 from m, $\mathrm{dist}(\sigma(0), P) \leq 1$. There are only 6 vectors of length 1 in $\mathbb{E}^2_\triangle$: $\pm e_1$, $\pm e_2$, $e_1 - e_2$, $e_2 - e_1$, hence the coordinates of $\sigma(0) = (a, b)$ satisfy: $a = -1$ and $1 \leq b \leq \beta + 1$ or $b = 1$ and $-\alpha - 1 \leq a \leq -1$ (since $a < 0 < b$).

In the first case m would contain $(0, k(b-1))$ for all $k \in \mathbb{N}$ – we proceed by induction: if the geodesic m passes through $(0, j(b-1))$, $1 \leq b \leq \beta + 1$, $j \geq 0$, then

$$\sigma((0, j(b-1))) = (-1, (j+1)(b-1) + 1)$$

is at distance 1 to m, what implies that m passes through $(0, (j+1)(b-1)))$. In the second case we similarly obtain that m would contain $(k(a+1), 0)$ for all $k \in \mathbb{N}$. In both cases we get contradiction with the assumption $\alpha, \beta > 0$.

Now we define an infinite simplicial complex S spanned by two convex geodesics in $\mathbb{E}^2_\triangle$ at distance 1 to each other to be a *thin strip* (it is shown in Figure 3.2). We especially are interested in simplicial maps $f : S \to X$ with thin strip S as a domain, such that the images of boundary geodesics of S are geodesics $g^i(l)$ and $g^j(l)$, where $0 \leq i < j < n$. Since geodesics $g^i(l)$ and $g^j(l)$ are disjoint (otherwise by Lemma 3.6 we obtain a g^{j-i}-invariant geodesic, contradicting the minimality of n), the map f is injective, so we may assume S is a subcomplex of X. If such a map f exists, we say that $g^i(l)$ and $g^j(l)$ span a thin strip.

Let $g^i(l)$ and $g^j(l)$, $0 \leq i < j < n$, span a thin strip S. The subcomplex $S' \subset S$ consisting of all edges $e \subset S$ such that $e \not\subset \partial S$ is combinatorically equivalent to a line. Thus it determines a linear order $\prec_{ij}$ on vertices of $g^i(l) \cup g^j(l)$. We set $x \prec g^n(x)$ for some, hence for every $x \in g^i(l) \cup g^j(l)$.

Step 4: *If geodesics $g^i(l)$ and $g^j(l)$, $0 \leq i < j < n$ span a thin strip $S \subset X$, then S is a full subcomplex of X.*

Since boundary lines of S are geodesic and disjoint (Lemma 3.6), $S \subset X$ in not a full subcomplex if and only if there are vertices a and x in different components of ∂S such that $d_S(a, x) = 2$ and $d_X(a, x) = 1$ (as in Figure 3.2).

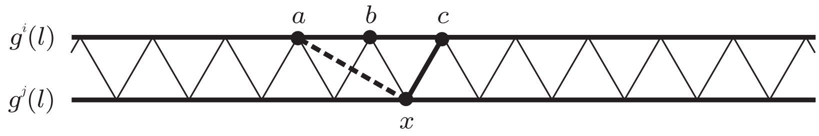

Figure 3.2.

Boundary geodesics $g^i(l)$ and $g^j(l)$ are g^n-invariant and g^n acts on them by translation of length ξ. Thus g^{2n} acts by translation of length $2\xi \geq 2$. If we replace vertices $g^{2\alpha n}(b) \in g^i(l)$ with $g^{2\alpha n}(x) \in g^j(l)$, $\alpha \in \mathbb{Z}$ we obtain a g^{2n}-invariant geodesic l'. Since $x \in g^j(l)$, $g^{i-j}(x) \in g^i(l)$ and $g^{i-j+n}(x) \in g^i(l)$. Vertices $g^{i-j}(x)$ and $g^{i-j+n}(x)$ are not in the same g^{2n}-orbit. As $g^i(l) \setminus l'$ contains only one g^{2n}-orbit, l' passes either through x and $g^{i-j}(x)$ or through x and $g^{i-j+n}(x)$. By Lemma 3.6 there exist g^{j-i}-invariant geodesic or g^{i-j+n}-invariant geodesic, contradicting the minimality of n.

Step 5: *If $g^i(l)$ and $g^j(l)$ span a thin strip and so do $g^j(l)$ and $g^k(l)$, $0 \leq i < j < k < n$, then $g^i(l)$ and $g^k(l)$ span a thin strip. Moreover, the relation $\prec_{ijk} = \prec_{ij} \cup \prec_{jk} \cup \prec_{ik}$ is transitive.*

Denote strips spanned by $g^i(l)$, $g^j(l)$ and by $g^j(l)$, $g^k(l)$ by S and S', respectively. By gluing strips $g^{\alpha(k-i)}(S)$ and $g^{\alpha(k-i)}(S')$ for $\alpha \in \mathbb{Z}$ we obtain a simplicial map $P : \mathbb{E}^2_\triangle \to X$ (as in Figure 3.3). By Lemma 3.6 the map P is nondegenerate, but it is not injective, as $S = g^{n(k-i)}(S)$.

If P restricted to $N(v)$ for any vertex $v \in \mathbb{E}^2_\triangle$ was an isometric embedding, then by Theorem 2.5 P would be an isometric embedding, contrary to the fact it is not injective. Thus there is a vertex $v \in \mathbb{E}^2_\triangle$ such that $P|_{N(v)}$ is not an isometric embedding. It suffices to check two cases: $v = a_0$ and $v = w_0$.

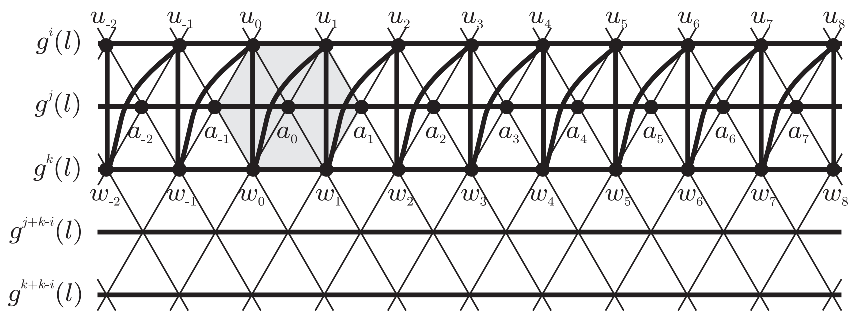

Figure 3.3.

The map $P|_{N(a_0)}$ is injective, since $P|_S$ and $P|_{S'}$ are injective and geodesics $g^i(l)$ and $g^k(l)$ are disjoint (Lemma 3.6). If $P|_{N(a_0)}$ is not an isometric embedding, there are two vertices in $\partial N(a_0)$ not connected by an edge in $\mathbb{E}^2_\triangle$, whose images are connected by an edge. Then, by systolicity of X, the image of $\partial N(a_0)$ can be filled with 4 triangles. Since by Step 4 $P(a_{-1})$ is not connected by an edge either with $P(u_1)$ or with $P(w_1)$ and $P(a_1)$ is not connected by an edge either with $P(u_0)$ or with $P(w_0)$, the filling is contains edges: $P(u_0)P(w_0)$, $P(u_1)P(w_1)$ and either $P(w_0)P(u_1)$ or $P(u_0)P(w_1)$. Without losing generality, we may assume the last edge is $P(w_0)P(u_1)$.

By induction on j we prove that there are edges $P(u_j)P(w_j)$ and $P(w_j)P(u_{j+1})$ in X: If there are edges $P(u_j)P(w_j)$ and $P(w_j)P(u_{j+1})$ then by systolicity the quadrilateral $P(w_j)P(u_{j+1})P(a_{j+1})P(w_{j+1})$ has a diagonal and by Step 4 it is $P(u_{j+1})P(w_{j+1})$; then again by systolicity of X the pentagon $P(w_{j+1})P(u_{j+1})P(u_{j+2})P(a_{j+2})P(w_{j+2})$ can be filled with 3 triangles; thus it has a diagonal $P(u_{j+2})P(w_{j+2})$ and either $P(u_{j+1})P(w_{j+2})$ or $P(u_{j+2})P(w_{j+1})$; however, the case $P(u_{j+1})P(w_{j+2})$ is impossible, since then proceeding as in Step 4 we would obtain contradiction with minimality of n. Thus $g^i(l)$ and $g^k(l)$ span a thin strip. The case $v = w_0$ is analogous.

The fact that $\prec_{ijk}$ is transitive is now clear from Figure 3.3.

Step 6: *There is a thick geodesic embedded into X.*

Denote $Y = \bigcup_{i=0}^{n-1} g^i(l)$. By Step 3 geodesics l and $g(l)$ span a thin strip (since m is a convex geodesic in $\mathbb{E}^2_\triangle$ and $\sigma(m)$ is 1-close to m). Thus, by Step 5 (proceeding by induction on k) geodesics l and $g^k(l)$, $0 < k < n$ span a thin strip. It follows that $g^i(l)$ and $g^j(l)$ span a thin strip for $0 \leq i < j < n$. By Step 4 $Y \subset X$ is a full subcomplex. We can introduce on the set $Y^{(0)}$ a relation

$$\prec = \bigcup_{0 \leq i < j < n} \prec_{ij}$$

which by Step 5 is well-defined and transitive. It also reflexive, antysymmetric and linear, as $\prec_{ij}$ is such. Hence $\prec$ defines a linear order on $Y^{(0)}$. Thus we may identify $Y^{(0)}$ with $\mathbb{Z}$. Since for any two consecutive vertices $a, a' \in g^i(l)$ and any $j \neq i$ there is exactly one vertex $b \in g^i(l)$ such that $a \prec b \prec a'$, the vertices of $g^i(l)$ are identified with $kn + r_i$, $k \in \mathbb{Z}$ for some r_i. It follows that vertices $a \in g^i(l)$ and $b \in g^j(l)$ are connected by an edge in X if and only if they are identified with integers of difference $\leq n$. Hence, the flag completion of Y is a g-invariant thick geodesic in X. □

Corollary 3.10. *Let G be a group acting properly discontinuously and cocompactly on some systolic complex X (a systolic group). Then any finitely generated abelian subgroup of G is undistorted.*

This fact is a consequence of Theorem 13.1 in [JS1], stating that systolic groups are biautomatic ([JS1], Theorem 13.1). Below we present, as an application of the theory of minimal surfaces, an alternative proof.

Proof: As a finitely generated abelian group has a finite-index free abelian subgroup, it suffices to prove that free abelian subgroups are non-distorted. In Theorem 6.1 in [E] we proved this for $\mathbb{Z}^n$, $n \geq 2$ (actually, it also states that the case $n > 2$ is impossible).

The case $n = 1$ follows from Theorem 3.5: Let X be a systolic complex admitting cocompact and properly discontinuous action of G and $x \in X$ an arbitrary vertex. Then the map $g \mapsto g(x)$ defines a quasi-isometry $q : G \to X$. By Theorem 3.5 for any infinite cyclic subgroup $H < G$ there exists an H-invariant quasi-geodesic l in X, which is mapped by the quasi-inverse of q to a quasi-geodesic k in G, such that $k \subset G$ and $H \subset G$ are at finite Hausdorff distance. Thus $H \hookrightarrow G$ is a quasi-isometric embedding, i.e. H is undistorted. □

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