Systolic groups acting on complexes with no flats are word–hyperbolic

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Abstract

We prove a theorem conjectured by D.T. Wise in [9], that if a group acts properly discontinuously and cocompactly on a systolic complex, in whose 1–skeleton there is no isometrically embedded copy of the 1–skeleton of an equilaterally triangulated Euclidean plane, then the group is word–hyperbolic.

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1 Introduction

Systolic complexes were introduced by J. Świątkowski and T. Januszkiewicz in [6] and independently by F. Haglund in [4]. They are simply–connected simplicial complexes satisfying certain link conditions. Their properties are very similar to the properties of CAT(0) metric spaces, therefore one calls them complexes of simplicial nonpositive curvature. In particular it was shown in [6] that they are contractible.

Geodesics (directed) are well defined for systolic complexes and one also has the notion of convexity. This was used by the authors of [6] to prove the theorem, that if a group Γ acts properly discontinuously and cocompactly by simplicial automorphisms on a systolic complex, then Γ is biautomatic, so also semihyperbolic. It was shown, that if one imposes a little stronger condition on links, the complex must be a hyperbolic metric space in the sense of Gromov (for the definition see [3]). A systolic complex does not have to be hyperbolic in general, for example equilaterally triangulated Euclidean plane is a two dimensional systolic complex. We prove that this is the only obstruction. Our result, which we formulate slightly later, is similar in spirit to the following well known theorem.

Theorem 1.1 ([1]). *If a group* Γ *acts properly and cocompactly by isometries on a locally compact CAT(0) space* X*, then* Γ *is word–hyperbolic if and only if* X *does not contain an isometrically embedded copy of the Euclidean plane.*

Since not every systolic complex is a $CAT(0)$ space, our goal is to prove a theorem which is a systolic analogue to Theorem 1.1:

Theorem 1.2. *Let* Γ *be a systolic group acting on a systolic complex* X*. Then* Γ *is word–hyperbolic if and only if there is no isometric embedding of the 1–skeleton of an equilaterally triangulated Euclidean plane into the 1–skeleton* $X^{(1)}$ *of* X*.*

After writing the proof it was communicated to us that an alternative version of proof could be constructed from the theorem of D.T. Wise [9] on minimal area embedded flat plane and from recent study by T. Elsner [2] on minimal flat surfaces in systolic complexes. Our proof, however, is more direct.

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2 Some information on systolic complexes

Let us recall the definition of a systolic complex and a systolic group following J. Świątkowski and T. Januszkiewicz [6].

Definition 2.1. A subcomplex K of a simplicial complex X is called *full* in X if any simplex of X spanned by vertices of K is a simplex of K. A simplicial complex X is called *flag* if any set of vertices, which are pairwise connected by edges of X, spans a simplex in X. A flag simplicial complex X is called *k–large*, $k \geq 4$ if there are no embedded cycles of length $< k$ being full subcomplexes of X.

Definition 2.2. A simplicial complex X is called *systolic* if it is connected, simply–connected and links of all simplices in X are 6–large. A group Γ is called *systolic* if it acts cocompactly and properly by simplicial automorphisms on a systolic complex X. (*Properly* means X is locally finite and for each compact subcomplex $K \subset X$ the set of $\gamma \in \Gamma$ such that $\gamma(K) \cap K \neq \emptyset$ is finite.)

Recall [6], that systolic complexes are themselves 6–large. In particular they are flag. Now we will briefly treat the definitions and facts concerning convexity:

Definition 2.3. For every pair of vertices A, B in a simplicial complex X denote by $|AB|$ the combinatorial distance between A, B in $X^{(1)}$, the 1–skeleton of X. A subcomplex K of a simplicial complex X is called *3–convex* if it is a full subcomplex of X and for every pair of edges AB, BC such that $A, C \in K, |AC| = 2$, we have $B \in K$. A subcomplex K of a systolic complex X is called *convex* if it is connected and links of all simplices in K are 3–convex subcomplexes of links of those simplices in X.

In chapter 8 of [6] authors conclude that convex subcomplexes of a systolic complex X are contractible, full and 3–convex in X. Now define the combinatorial ball $B_n(Y) = \text{span}\{P \in X : |PS| \le n \text{ for some vertex } S \in Y\}$, where $n \ge 0, Y \subset X$. If Y is convex (in particular, if Y is a simplex) then $B_n(Y)$ is a convex subcomplex of a systolic complex X, as proved in [6].

We will need a crucial projection Lemma 14 of [6], which we will apply in most cases to σ being edges. Define the *residue*of a simplex σ in X as the union of all simplices in X, which contain σ.

Lemma 2.4. *Let Y be a convex subcomplex of a systolic complex X and let σ be a simplex in $B_1(Y)\backslash Y$. Then the intersection of the residue of σ and of the complex Y is a simplex (in particular it is nonempty).*

Definition 2.5. The simplex as in Lemma 2.4 is called the *projection* of σ onto Y.

Now for a pair of vertices V, W, $|VW| = n$ in a systolic complex X we define inductively a series of simplices $\sigma_0 = V, \sigma_1, \ldots, \sigma_n = W$ as follows. Take σ_{i+1} equal to the projection of σ_i onto $B_{n-1-i}(W)$ for $i = 0, 1, \ldots, n-1$. The series (σ_n) is called the *directed geodesic* from V to W. It is proved in [6] that if $V, W \in K$ which is a convex subcomplex of a systolic complex X, then the the simplices of the directed geodesic from V to W (and also from W to V) are all in K. We will need this in the form of the following corollary.

Corollary 2.6. *For every pair of vertices V, W in a systolic complex X there exists a 1–skeleton geodesic connecting V to W, such that if V, W belong to a common convex subcomplex K of X, then this geodesic is also contained in K.*

Definition 2.7. We will call a 1–skeleton geodesic satisfying Corollary 2.6 a *special* geodesic.[1]

[1]We were informed that J. Świątkowski and F. Haglund have proved in [5] that every 1–skeleton geodesic in a systolic complex is special in this sense.

3 Embedding lemmas

In this section we prepare the proof of the main theorem.

Definition 3.1. A two dimensional simplicial complex with distinguished vertices A, B and C is called a *k–triangle ABC*, $k \geq 0$ if it is simplicially equivalent to equilateral triangulation into k^2 simplices of an Euclidean triangle of edge length k, with vertices A, B, C corresponding to the vertices of the original Euclidean triangle.

Lemma 3.2. *Let $D\colon \Delta \to X$ be a simplicial mapping from Δ, a k–triangle ABC, into a systolic complex X, such that for any vertex $V \in \{A, B, C\}$ and any vertex P lying in Δ on a geodesic connecting the two other vertices from the set $\{A, B, C\}$ we have $|D(V)D(P)| = k$. Then D considered as mapping between 1–skeletons of Δ and X is an isometric embedding.*

Proof. Take any different vertices $R, S \in \Delta$. We claim that R, S lie on a certain 1–skeleton geodesic in Δ connecting a vertex $V \in \{A, B, C\}$ to some point P defined as in the hypothesis of the lemma. This can be observed in the following way. Recall that k–triangle Δ carries Euclidean structure. Consider three straight Euclidean lines going through R contained in the 1–skeleton of Δ. They divide Δ into six regions. Now, depending on which region is vertex S in, it is easy to point out vertices V, P and a geodesic VP containing R and S. (Vertices V, P belong to the sector S is in and to the opposite sector.) By the hypothesis of the lemma D must embed geodesic VP into $X^{(1)}$, so it also preserves the 1–skeleton distance between R and S. This means, that D considered as mapping between 1–skeletons of Δ and X is an isometric embedding. □

Lemma 3.3. *Let $D\colon \Delta \to X$ be a simplicial mapping from Δ, a k–triangle ABC, into a systolic complex X, such that $|D(A)D(B)| = |D(B)D(C)| = |D(C)D(A)| = k$. Denote by AB the unique length n path in Δ between vertices A, B consisting of k edges and $k+1$ vertices. If there exists a convex $Z \subset X$ such that $D^{-1}(B_l(Z)) = B_l(AB)$ for $l = 0, 1, \ldots, k$ then D considered as mapping between 1–skeletons of Δ and X is an isometric embedding.*

Proof. Note that the hypothesis immediately implies that the distance between $D(C)$ and $D(AB)$ is equal k. In order to apply Lemma 3.2 we have to prove the same for $D(B), D(AC)$ and $D(A), D(BC)$. We focus on the last pair.

Denote by P_i^j the unique vertex of Δ which lies at distance i in the 1–skeleton of Δ from C and at distance j in the 1–skeleton of Δ from A, $0 \leq i, j \leq k,\ i + j \geq k$.

We will prove by backward induction that the distance $|D(P_i^k)D(A)| = |P_i^k A| = k$ for $i = k, k-1, \ldots, 0$. For $i = k$ we have $P_k^k = B$, so $|D(P_k^k)D(A)| = |D(B)D(A)| = k = |BA|$ is already an assumption of the lemma.

Suppose we have proved already the equality for all $i : 0 \leq s < i \leq k$. We will prove now the equality for $i = s$. Let $D(A) = S_0, S_1, \ldots, S_{m-1}, S_m = D(P_s^k)$ be consecutive vertices of a 1–skeleton special geodesic of length m joining $D(A)$ with $D(P_s^k)$ in X. Notice that S_m is at distance $k - s$ from Z, but S_0 belongs to Z. Assume $r < m$ is the biggest number such that $S_r \in B_{k-s-1}(Z)$. Due to convexity of balls the vertices $S_q,\ m \geq q > r$ belong to $B_{k-s}(Z)$. Now for each edge $S_q S_{q+1},\ r < q < m$ choose a point R_q in $B_{k-1-s}(Z)$ contained in the projection of $S_q S_{q+1}$ onto $B_{k-1-s}(Z)$. By the projection properties (Lemma 2.4) the sequence of vertices $D(A) = S_0, S_1, \ldots, S_r, R_{r+1}, R_{r+2}, \ldots, R_{m-1}, D(P_{s+1}^k)$ is connected by edges in the 1–skeleton of X and therefore by induction hypothesis we have $m \geq k$. By choosing a path in X between $D(A)$ and $D(P_s^k)$ which is an image of geodesic path between A and P_s^k in Δ one sees that $|D(A)D(P_s^k)| \leq k$, so altogether $|D(A)D(P_s^k)| = k$, which is the required induction step equality.

In this way we have proved that the distance between $D(A)$ and $D(BC)$ is equal k. By repeating the same argument we obtain also that for any $i, j \geq 0, i + j = k$ we have $|D(B)D(P_i^j)| = k$. Now we know, that the distance in $X^{(1)}$ between vertices $D(A), D(B), D(C)$ and vertices which are images of the opposite edges in k–triangle Δ are all equal to k, so we can apply Lemma 3.2. □

Lemma 3.4. *Let Γ be a group acting cocompactly on a locally finite systolic complex X. If for arbitrarily large $n > 0$ there exists an isometric embedding of the 1–skeleton of a n–triangle Δ into $X^{(1)}$, then there exists an isometric embedding of the 1–skeleton of equilaterally triangulated Euclidean plane into $X^{(1)}$.*

Proof. Denote by E the equilaterally triangulated Euclidean plane and by Δ_0 any vertex of E. For all $k \geq 0$ pick k–triangles $\Delta_k \subset E$ such that $\Delta_k \subset \Delta_{k+1}$ and $\bigcup_{k=0}^{\infty} \Delta_k = E$.

We will define inductively isometric embeddings $f_k \colon \Delta_k^{(1)} \to X^{(1)}$ such that $f_{k+1|\Delta_k^{(1)}} = f_k$. The union $\bigcup_{k=0}^{\infty} f_k \colon E^{(1)} \to X^{(1)}$ will be the desired isometric embedding.

First, the hypothesis of the lemma guarantees that for arbitrarily large n there exist embeddings $D_n \colon \Delta_n^{(1)} \to X^{(1)}$. Since Γ acts cocompactly on X, we can choose $\gamma_n \in \Gamma$ such that $\gamma_n \circ D_n(\Delta_0)$ belongs to a finite set of vertices in X. By passing to a subsequence and substituting D_n with $\gamma_n \circ D_n$ we can assure that $D_n(\Delta_0)$ does not depend on n. We then define $f_0 \colon \Delta_0 \to X^{(1)}$ by $f_0(\Delta_0) = D_n(\Delta_0)$.

Now suppose we have already defined an embedding $f_k \colon \Delta_k^{(1)} \to X^{(1)}$. Note that $\Delta_{k+1}^{(1)} \backslash \Delta_k^{(1)}$ is finite and $B_1(\mathrm{Im}(f_k))$ is also finite (because X is locally finite), so by passing to a subsequence we can assure that $D_{n|\Delta_{k+1}^{(1)}}$ does not depend on n. We then define $f_{k+1} \colon \Delta_{k+1}^{(1)} \to X^{(1)}$ by $f_{k+1} = D_{n|\Delta_{k+1}^{(1)}}$. This ends the induction step. □

4 Hyperbolicity

We are ready to prove the main theorem of the paper.

Proof of Theorem 1.2. The implication from left to right is easy. If $X^{(1)}$, the 1–skeleton of a systolic complex X, contains an isometrically embedded 1–skeleton of the triangulated Euclidean plane then $X^{(1)}$ is not a hyperbolic metric space, so Γ is not word–hyperbolic.

Now we will prove right to left implication. Suppose Γ is not word–hyperbolic. Then, by a theorem of P. Papasoglu [8] bigons in $X^{(1)}$are not thin, i.e. for every $n \in \mathbb{N}$ there exist vertices $V, Y \in X$ and two 1–skeleton geodesics R, S, joining V, Y (denote their consecutive vertices by $V = R_0$, $R_1, \ldots, R_{m-1}, R_m = Y; V = S_0, S_1, \ldots, S_{m-1}, S_m = Y$) and there exists $t : 0 < t < m$, such that $|R_t S_t| > n$. Denote $k = |R_t S_t| > n$, choose a special 1–skeleton geodesic of length k connecting R_t, S_t and denote its consecutive vertices by $R_t = P_k^0, P_k^1, \ldots, P_k^{k-1}, P_k^k = S_t$. Now construct inductively vertices $P_i^j \in X,\ 0 \leq i, j \leq k,\ i + j \geq k$ in the following way. For $i = k$ the vertices are already given. Suppose we have already constructed vertices P_i^j for all i such that $p < i \leq k$, where i, j are as above. Now we will define vertices P_i^j for $i = p$. For each j such that $k - p \leq j \leq k$ project the edge $P_{p+1}^{j-1} P_{p+1}^j$ onto the ball $B_{t-(k-p)}(V)$ and denote any vertex of this projection by P_p^j.

Now notice that for a fixed l, such that $0 \leq l \leq k$, the vertices P_i^j such that $i \geq k - l$ are all contained in the ball $B_{m-t+l}(Y) = B_l(B_{m-t}(Y))$ and no other vertex P_i^j belongs to this ball. This means that the k–triangle formed by vertices P_i^j satisfies all the assumptions of Lemma 3.3 with $Z = B_{m-t}(Y), D$ being identity and therefore the 1–skeleton of this k–triangle is isometrically embedded in $X^{(1)}$. Since $k > n$ can be chosen arbitrarily large, the hypothesis of Lemma 3.4 is satisfied and we obtain the 1–skeleton of the equilaterally triangulated Euclidean plane isometrically embedded in $X^{(1)}$. □

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