Fundamental pro-groups and Gromov boundaries of 7-systolic groups

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Abstract. We introduce the property of pro-π_1-saturation (defined in terms of fundamental pro-groups) for compact metric spaces. We expect (though not yet prove) this property to be stronger than hereditary asphericity. We show that 1-dimensional spaces and Gromov boundaries of 7-systolic groups are pro-π_1-saturated (the latter class contains examples of pro-π_1-saturated spaces with arbitrary finite topological dimension).

1. Introduction

Systolic complexes were introduced in [JS], as simply connected simplicial complexes which satisfy certain local combinatorial condition that mimicks nonpositive curvature. 7-systolic complexes satisfy slightly stronger local condition, and share many properties with negatively curved spaces, e. g. they are hyperbolic in the sense of Gromov. 7-systolic groups are ones that act geometrically on 7-systolic complexes. In particular, they are word-hyperbolic. It was shown in [JS] that 7-systolic groups do exist in arbitrary cohomological dimension, and thus that their Gromov boundaries have arbitrary topological dimension.

Damian Osajda has shown in [O] that Gromov boundaries of 7-systolic groups are strongly hereditarily aspherical. (Hereditarily aspherical spaces were introduced by R. Davermann in [D], for the purpose of getting spaces for which the cell-like maps do not raise dimension.) To prove his result, Osajda describes Gromov boundary ∂X of a 7-systolic complex X as inverse limit of the sequence of combinatorial spheres in X, with some appropriately defined bonding maps. We will refer to this inverse sequence of spheres as *Osajda's inverse sequence*. The result of Osajda excludes spheres and Menger compacta in dimensions ≥ 2 (and many other topological spaces) from being Gromov boundaries of 7-systolic groups. In fact, in high dimensions 7-systolic groups are the only known word hyperbolic groups with hereditarily aspherical Gromov boundary.

In this paper, by taking a closer look at Osajda's inverse sequence, we isolate another property of Gromov boundaries ∂X of 7-systolic complexes X. We name this property *pro-π_1-saturation*. It says that, for any closed subset $Y \subset \partial X$ and any point $z_0 \in Y$ the morphism of fundamental pro-groups i_*^{pro} : pro-$\pi_1(Y, z_0) \to$ pro-$\pi_1(\partial X, z_0)$ induced by the inclusion $i : Y \to \partial X$ is a monomorphism in the category of pro-groups. The statement of our main result is as follows.

Main Theorem. *Let X be a locally finite 7-systolic simplicial complex. Then its Gromov boundary ∂X is pro-π_1-saturated. In particular, Gromov boundary of any 7-systolic group is pro-π_1-saturated.*

An explicit rich class of groups to which the result above applies, containing examples with arbitrary (virtual) cohomological dimension, is that of right-angled Coxeter groups whose nerves are 7-large simplicial complexes (the term 7-large is explained in Definition 3.1). For more details see Comment 9 in Section 8.

It seems to us that the property of pro-π_1-saturation is stronger than that of hereditary asphericity, though we don't know how to prove this. Clearly, there are hereditarily aspherical spaces that are not pro-π_1-saturated, e.g. a 2-disc (see Remark 2.2(1)). We observe that all spaces of topological dimension 1 are pro-π_1-saturated (see Proposition 2.7). It is not clear if the notions of hereditary asphericity and pro-π_1-saturation are different in the class of Gromov boundaries of word-hyperbolic groups. All this rather brings new questions than solves any open problems.

The paper is organized as follows. In Section 2 we recall basic definitions and facts related to fundamental pro-groups, and introduce the property of pro-π_1-saturation. We also describe a weaker variant of this property related to shape fundamental groups. We show that 1-dimensional compact metric spaces are pro-π_1-saturated, and discuss examples and non-examples in higher dimensions.

In Section 3 we recall, mainly from [JS], the necessary material concerning systolic and 7-systolic complexes and groups. In Section 4 we describe, after [O], the Osajda's inverse sequence for a 7-systolic complex.

Sections 5 and 6 contain new observations concerning the Osajda's inverse sequence. They allow to prove, in Section 7, the Main Theorem. Section 5 contains description of the dual cellulations of simplicial complexes, and an observation that the preimage through the bonding map $p_n : S_{n+1}(v_0, X) \to S_n(v_0, X)$ of any subcomplex Q for the dual cellulation of $S_n(v_0, X)$ is π_1-injectively included in the sphere $S_{n+1}(v_0, X)$. In Section 6 we express any closed subset $Y \subset \partial X$ as inverse limit of some sequence of subcomplexes $Y_n \subset S_n(v_0, X)$, so that each Y_n is π_1-injectively included in the corresponding sphere. We do this using the result of Section 5.

Finally, in Section 8 we make some further comments, ask questions and formulate conjectures.

2. Pro-π_1- and shape π_1-saturated spaces.

In this section we recall the notions of fundamental pro-group and shape fundamental group, and introduce two variants of the property of π_1-saturation corresponding to those two notions. Our main reference with the appropriate background is [MS].

Any pointed compact metric space (Z, z_0) can be expressed as the inverse limit of a sequence (Z_n, z_n) of pointed compact polyhedra equipped with PL bonding maps

$$q_{n+1,n} : (Z_{n+1}, z_{n+1}) \to (Z_n, z_n).$$

We will call any such expression a *polyhedral expansion* of (Z, z_0). Associated to any such expansion, there is an inverse sequence

$$\pi_1(Z_n, z_n),\ (q_{n+1,n})_* : \pi_1(Z_{n+1}, z_{n+1}) \to \pi_1(Z_n, z_n),\ n \in \mathrm{N}$$

of fundamental groups. The *shape fundamental group* $\check{\pi}_1(Z, z_0)$ of Z at z_0 is defined as the inverse limit

$$\check{\pi}_1(Z, z_0) := \varprojlim \big(\pi_1(Z_n, z_n), (q_{n+1,n})_*\big).$$

The shape fundamental group is well defined, i.e. does not depend on the choice of a polyhedral expansion for (Z, z_0).

If Z is locally path connected and semilocally simply connected then its shape fundamental group coincides with the ordinary fundamental group. Otherwise, the groups are in general different. If Z is a Z-set boundary of an ANR X (e.g. when it is the Gromov boundary of a group), then the shape fundamental group coincides with the fundamental group at infinity of X. In this paper we will be interested in the situation when Z is the Gromov boundary of a 7-systolic group.

A continuous map $u : (Y, y_0) \to (Z, z_0)$ between metric compacta induces a well defined homomorphism $u_*^\vee : \check{\pi}_1(Y, y_0) \to \check{\pi}_1(Z, z_0)$. In particular, if Z is a compact metric space and $Y \subset Z$ is a closed subset then for any $z_0 \in Y$ the inclusion $i : (Y, z_0) \to (Z, z_0)$ induces the homomorphism $i_*^\vee : \check{\pi}_1(Y, z_0) \to \check{\pi}_1(Z, z_0)$.

Definition 2.1. A compact metric space Z is *shape* π_1*-saturated* (or $\check{\pi}_1$*-saturated*) if for any closed subset $Y \subset Z$ and any point $z_0 \in Y$ the corresponding homomorphism $i_*^\vee : \check{\pi}_1(Y, z_0) \to \check{\pi}_1(Z, z_0)$ induced by the inclusion is injective.

Remarks and examples 2.2.

(1) Note that if Z contains a 2-disc then it is not $\check{\pi}_1$-saturated. Indeed, in this case Z contains a circle S^1 such that the homomorphism $i_*^\vee : \check{\pi}_1(S^1, s_0) = \pi_1(S^1, s_0) \to \check{\pi}_1(Z, s_0)$ is trivial and thus not injective.

(2) By (1), a polyhedron H is $\check{\pi}_1$-saturated iff $\dim H = 1$.

(3) Spheres S^n and n-dimensional Menger compacta are not $\check{\pi}_1$-saturated for $n \geq 2$.

(4) It follows from the results of this paper, and a result of P. Zawiślak [Z], that the Pontriagin sphere and the Pontriagin surface Π_2 are $\check{\pi}_1$-saturated (they occur as Gromov boundaries of some 7-systolic groups). The direct proof of this fact is probably not hard.

(5) There exist $\check{\pi}_1$-saturated metric compacta with arbitrary (finite) topological dimension. This follows from the constructions in [JS] of 7-systolic groups of arbitrary dimension.

A more subtle information about metric compacta can be described in terms of pro-groups, i.e. inverse sequences of groups viewed up to some natural equivalence relation. We recall here only some basic facts related to this approach, referring the reader to [MS] for further details.

In the class inv-Group of inverse sequences of groups there is a natural notion of morphism. To describe it, for any $(G_n, h_{n+1,n}) \in$ inv-Group and any $n > k$ denote by $h_{n,k} : G_n \to G_k$ the composition homomorphism $h_{k+1,k} \circ \ldots \circ h_{n,n-1}$. If $(G_n, h_{n+1,n})$ and $(G'_n, h'_{n+1,n})$ are in inv-Group then a morphism

$$f : (G_n, h_{n+1,n}) \to (G'_n, h'_{n+1,n})$$

consists of a function $\phi : \mathrm{N} \to \mathrm{N}$ and of homomorphisms $f_n : G_{\phi(n)} \to G'_n$, one for each n, such that whenever $n_1 < n_2$ then there is $m > \max(\phi(n_1), \phi(n_2))$ for which

$$f_{n_1} h_{m,\phi(n_1)} = h'_{n_2,n_1} f_{n_2} h_{m,\phi(n_2)},$$

i.e. the following diagram commutes

$$\begin{array}{ccccc} G_{\phi(n_1)} & \leftarrow & G_m & \rightarrow & G_{\phi(n_2)} \\ \downarrow & & & & \downarrow \\ G'_{n_1} & & \longleftarrow & & G'_{n_2} \end{array}$$

Composition of morphisms in inv-Group is defined in the obvious way.

Two morphisms $f = (\phi, f_n)$ and $g = (\psi, g_n)$ from $(G_n, h_{n+1,n})$ to $(G'_n, h'_{n+1,n})$ are *equivalent* if for each n there is $m > \max(\phi(n), \psi(n))$ such that

$$f_n h_{m,\phi(n)} = g_n h_{m,\psi(n)}.$$

This defines an equivalence relation which respects compositions and thus allows to define a new category pro-Group as follows. The objects in pro-Group are all inverse sequences of groups and the morphisms are all equivalence classes of morphisms in inv-Group.

Given a pointed compact metric space (Z, z_0) and its arbitrary polyhedral expansion $((Z_n, z_n), q_{n+1,n})$, the induced inverse sequence of fundamental groups

$$\big(\pi_1(Z_n, z_n), (q_{n+1,n})_*\big)$$

is well defined up to an isomorphism in the category pro-Group. More precisely, different polyhedral expansions of (Z, z_0) yield induced inverse sequences of fundamental groups that are isomorphic in pro-Group. The *fundamental pro-group* pro-$\pi_1(Z, z_0)$ is the induced inverse sequence of fundamental groups of a polyhedral expansion of (Z, z_0), viewed up to isomorphism in pro-Group.

Note that, since isomorphic inverse systems of groups have isomorphic inverse limits, the fundamental pro-group pro-$\pi_1(Z, z_0)$ determines uniquely the shape fundamental group $\check{\pi}_1(Z, z_0)$. It is also (in general) a much more subtle invariant of (Z, z_0) than the latter.

As in the case of shape fundamental groups, a continuous map $u : (Y, y_0) \to (Z, z_0)$ induces a well defined up to equivalence (or as a morphism in pro-Group) morphism

$$u_*^{\text{pro}} : \text{pro-}\pi_1(Y, y_0) \to \text{pro-}\pi_1(Z, z_0).$$

A morphism $f : X \to Y$ in an arbitrary category is called a *monomorphism* provided $fg = fg'$ implies $g = g'$ for any morphisms $g, g' : W \to X$. We recall from [MS] (Corollary 1 on p. 108) a useful sufficient condition for a morphism in pro-Group to be a monomorphism.

Lemma 2.3. *Let F be a morphism in pro-Group given by a morphism*

$$f = (\phi, f_n) : (G_n, h_{n+1,n}) \to (G'_n, h'_{n+1,n})$$

in inv-Group, and suppose that $\phi = id_{\mathrm{N}}$ and that each $f_n : G_n \to G'_n$ is injective. Then F is a monomorphism.

We are now ready to introduce the notion of pro-π_1-saturation.

Definition 2.4. A compact metric space Z is *pro-π_1-saturated* if for any closed subset $Y \subset Z$ and any point $z_0 \in Y$ the morphism i_*^{pro} : pro-$\pi_1(Y, z_0) \to$ pro-$\pi_1(Z, z_0)$ induced by the inclusion $i : (Y, z_0) \to (Z, z_0)$ is a monomorphism in pro-Group.

Remarks 2.5.

(1) Any morphism in inv-Group induces a homomorphism between the corresponding inverse limits, and equivalent morphisms induce equal homomorphisms. In particular, a morphism in pro-Group induces a homomorphism between the corresponding inverse limits. Moreover, the homomorphism of the inverse limits induced by a monomorphism in pro-Group is injective.

(2) It follows from (1) that any compact metric space Z which is pro-π_1-saturated is also $\check{\pi}_1$-saturated.

(3) Remarks and examples 2.2 hold true if we replace $\check{\pi}_1$-saturation with pro-π_1-saturation.

Let $((Z_n, z_n), q_{n+1,n})$ be a polyhedral expansion of a pointed compact metric space (Z, z_0), and let $z_0 \in Y \subset Z$ be a closed subset of Z. It is possible to obtain a polyhedral expansion for (Y, z_0) out of a sequence of subpolyhedra $z_n \in Y_n \subset Z_n$, with the restrictions $h_{n+1,n}|_{Y_{n+1}} : Y_{n+1} \to Y_n$ as bonding maps. If we are given such expansion for (Y, z_0), then the morphism

$$i_*^{\text{pro}} : \text{pro-}\pi_1(Y, z_0) \to \text{pro-}\pi_1(Z, z_0)$$

induced by the inclusion is given by the natural morphism

$$f = (\phi, f_n) : \big(\pi_1(Y_n, z_n), (q_{n+1,n}|_{Y_{n+1}})_*\big) \to \big(\pi_1(Z_n, z_n), (q_{n+1,n})_*\big)$$

in inv-Group such that $\phi = id_{\mathrm{N}}$ and $f_n = (i_n)_* : \pi_1(Y_n, z_n) \to \pi_1(Z_n, z_n)$, where $i_n : (Y_n, z_n) \to (Z_n, z_n)$ are the inclusions. This description of the morphisms i_*^{pro} will be useful for establishing the property of pro-π_1-saturation.

We now formulate a characterization of sequences $Y_n \subset Z_n$ of subpolyhedra that give polyhedral expansions for a closed subspace $Y \subset Z$. It follows fairly directly from Theorem 6 on p. 60 in [MS].

Lemma 2.6. *Let (Z, z_0) be a pointed compact metric space, and let $((Z_n, z_n), q_{n+1,n})$ be its polyhedral expansion. Denote by $q_n : (Z, z_0) \to (Z_n, z_n)$ the canonical maps from the inverse limit (forming together the morphism associated to the inverse limit from (Z, z_0) to the inverse sequence). Let $z_0 \in Y \subset Z$ be a closed subset of Z, and let $z_n \in Y_n \subset Z_n$ be a sequence of subpolyhedra. Then the sequence $((Y_n, z_n), q_{n+1,n}|_{Y_{n+1}})$ is a polyhedral expansion for (Y, z_0) iff the following conditions are satisfied:*

1. *for each n we have $q_{n+1,n}(Y_{n+1}) \subset Y_n$,*
2. *for each n we have $q_n(Y) \subset Y_n$, and*
3. *for each n and for any open set U with $q_n(Y) \subset U \subset Z_n$ there is $m > n$ such that $q_{m,n}(Y_m) \subset U$ (where $q_{m,n} := q_{n+1,n} \circ q_{n+2,n+1} \circ \ldots \circ q_{m,m-1}$).*

Condition (1) in above lemma ensures that $((Y_n, z_n), q_{n+1,n}|_{Y_{n+1}})$ is an inverse sequence. Condition (2) means that $(q_n|_Y)$ is a morphism from Y to this sequence. Condition (3) guarantees that $(q_n|_Y)$ coincides with the morphism associated to the inverse limit.

In Section 6 we will use Lemma 2.6 for constructing certain polyhedral expansions for closed subsets in Gromov boundaries of 7-systolic complexes.

We finish the section with the following result, the proof of which exhibits usefulness of Lemmas 2.3 and 2.6 for showing that a space is pro-π_1-saturated.

Proposition 2.7. *Any compact metric space X of topological dimension* $\dim X = 1$ *is pro-π_1-saturated.*

Proof: If $\dim X = 1$ and x_0 is any point of X, then (X, x_0) has a polyhedral expansion $\big((X_n, x_n), (q_{n+1,n})\big)$ in which all X_n are graphs (see Theorem 3 on p. 90 or Remark 2 on p. 91 in [MS]). If Y is a closed subspace of X with $X-) \in Y$, one can easily choose a sequence of sub-polyhedra (i.e. subgraphs for appropriately subdivided graph structures) $Y_n \subset X_n$, with $x_n \in Y_n$, satisfying conditions (1)–(3) of Lemma 2.6. By Lemma 2.6, $\big((Y_n, x_n), (q_{n+1,n})|_{Y_{n+1}}\big)$ is then a polyhedral expansion for Y.

Let $i : Y \to X$ and $i_n : Y_n \to X_n$ be the inclusions. By the comment in the paragraph after Remarks 2.5, the morphism i_*^{pro} : pro-$\pi_1(Y, x_0) \to$ pro-$\pi_1(X, x_0)$ is given by the natural morphism

$$f = (\phi, f_n) : \big(\pi_1(Y_n, x_n), (q_{n+1,n}|_{Y_{n+1}})_*\big) \to \big(\pi_1(X_n, x_n), (q_{n+1,n})_*\big)$$

in inv-Group such that $\phi = id_{\mathrm{N}}$ and $f_n = (i_n)_* : \pi_1(Y_n, x_n) \to \pi_1(X_n, x_n)$.

Now, since Y_n is a subgraph in X_n, the homomorphisms $(i_n)_*$ above are injective. By Lemma 2.3, the morphism i_*^{pro} is then a monomorphism, which completes the proof.

3. Systolic complexes and groups.

In this section we recall basic definitions and facts concerning systolic complexes and groups, the objects studied in a recently introduced subject of simplicial nonpositive curvature. Our main reference for this subject is [JS].

Definition 3.1 (k-large complex). Given an integer $k \geq 5$, a simplicial complex L is *k-large* if it is flag, and if every full subcomplex of L homeomorphic to the circle S^1 contains at least k edges. (Here *flagness* of L means that any finite set of vertices of L pairwise connected with edges spans a simplex of L.)

We use k-largeness, by applying it to links in a simplicial complex, as a sort of local curvature bound. Recall that a *link* of a simplicial complex X at its simplex σ, denoted X_σ, is a subcomplex of X consisting of all simplices τ disjoint with σ and such that τ and σ span a simplex of X.

Definition 3.2 (k-systolic complex and group). A simplicial complex X is *k-systolic* if it is connected, simply connected and for any simplex σ the link X_σ is k-large. A group is k-systolic if it acts geometrically (i.e. properly discontinuously and cocompactly) by simplicial automorphisms on some k-systolic complex. The case $k = 6$ is the most important one and thus we abbreviate 6-systolic to *systolic*.

Systolic complexes turned out to be good combinatorial analogs of nonpositively curved (or CAT(0)) geodesic metric spaces. Though no way of reducing systolicity to CAT(0) (for appropriately chosen metric) is known, it is established that systolic complexes are contractible ([JS], Theorem B) while systolic groups are biautomatic and thus semihyperbolic ([JS], Theorem E). Systolic complexes of groups are known to be developable ([JS], Theorem D).

If $k > 6$, the condition of k-systolicity is stronger, and corresponds to some kind of negative curvature. In particular, 7-systolic complexes and groups are known to be hyperbolic in the sense of Gromov ([JS], Theorem A).

Developability of systolic complexes of groups allowed to construct (in [JS]) numerous examples of k-systolic complexes and groups, for any k and in arbitrary dimension. For $k \geq 7$ this gave examples of high dimensional word hyperbolic groups with unexpected properties. For example, these groups contain no subgroups isomorphic to fundamental groups of nonpositively curved closed manifolds of dimension ≥ 3, see Theorem A in [JS2]. Furthermore, by a result of D. Osajda [O], Gromov boundaries of these groups satisfy a rather restricted property of strong hereditary asphericity. Both properties above do not hold for the known word hyperbolic groups of dimensions above 2, except the systolic ones. The main result of this paper establishes another such property - pro-π_1-saturation of the Gromov boundary.

We now turn to listing some basic properties of large and systolic complexes needed later in this paper. We start with the following easy fact (see [JS], Fact 1.2(2)).

Lemma 3.3. *Any full subcomplex of a k-large simplicial complex is k-large.*

The next property is a nontrivial fact proved in [JS] as Proposition 1.4.

Lemma 3.4. *For $k \geq 6$, any k-systolic simplicial complex is k-large.*

Combining the above two lemmas we get the following.

Corollary 3.5. *For $k \geq 6$, any full subcomplex of a k-systolic simplicial complex is k-large.*

Since links in flag simplicial complexes are full subcomplexes, we also get the following.

Corollary 3.6. *Link X_σ at any simplex σ in a 6-large simplicial complex X is itself 6-large. In particular, this holds true if X is systolic.*

Further properties of k-large simplicial complexes are related to the following concept of 3-convexity. A subcomplex K in a simplicial complex L is 3-*convex* if it is full and for any geodesic path γ in the 1-skeleton of L, connecting vertices of K lying at polygonal distance 2 in L, the mid vertex of γ also belongs to K. A subcomplex $K \subset L$ is *locally* 3-*convex* if for any simplex σ of K the link K_σ is 3-convex in the corresponding link L_σ.

The following property is fairly straightforward (see [JS], Fact 3.3(1)).

Lemma 3.7. *Any 3-convex subcomplex in a flag simplicial complex L is locally 3-convex. In particular, this holds true if L is k-large.*

The property of π_1-saturation for boundaries of 7-systolic complexes and groups will be derived from the following π_1-injectivity property of locally 3-convex subcomplexes. This highly notrivial fact is a special case of Theorem 4.1(2) in [JS].

Proposition 3.8. *Let S be a 6-large simplicial complex and let Q be its locally 3-convex subcomplex. Then Q is π_1-injective in S. More precisely, for any point $q \in Q$ the homomorphism $i_* : \pi_1(Q, q) \to \pi_1(X, q)$ induced by the inclusion $i : Q \to S$ is injective.*

Next properties that we need concern combinatorial balls and spheres in systolic complexes.

Definition 3.9 (combinatorial ball). Let X be a simplicial complex and σ its any simplex. For integer $n \geq 0$, *combinatorial balls* $B_n(\sigma, X)$ centered at σ are the subcomplexes of X defined recursively as follows: $B_0(\sigma, X) := \sigma$ and $B_{n+1}(\sigma, X)$ is the union of all simplices of X that intersect $B_n(\sigma, X)$.

Clearly, the vertex set of the ball $B_n(\sigma, X)$ coincides with the set of all vertices in X remaining at polygonal distance from σ (in the 1-skeleton of X) $\leq n$. However, in general the ball needn't to be the full subcomplex spanned on this set of vertices. In systolic complexes the balls behave very nicely. The following property is extracted from Corollary 7.5 and Lemma 7.6 in [JS].

Lemma 3.10. *Any ball $B_n(\sigma, X)$ in a systolic simplicial complex X coincides with the full subcomplex spanned by the set of all vertices of X at polygonal distance from σ (in the 1-skeleton of X) $\leq n$. Moreover, every ball is locally 3-convex in X.*

Next result is implicitly present but nowhere explicitly stated in [JS], and thus we include its proof.

Lemma 3.11. *Any ball $B_1(\sigma, X)$ in a 6-large simplicial complex X is 3-convex in X.*

Proof: Note that, since X is 6-large, the shortest homotopically notrivial polygonal path in X has length at least 6. It follows that $B_1(\sigma, X)$ is isomorphic with $B_1(\tilde{\sigma}, \widetilde{X})$, where $\widetilde{X}$ is the universal cover of X and $\tilde{\sigma}$ is a lift of σ. Now, since X is locally 6-large (Corollary 3.6), the same is true for $\widetilde{X}$, and hence the latter is systolic. By Lemma 3.10, $B_1(\tilde{\sigma}, \widetilde{X})$ is then locally 3-convex in $\widetilde{X}$, and hence the same is true for $B_1(\sigma, X)$ in X. A similar argument shows that, since $B_1(\tilde{\sigma}, \widetilde{X})$ is full in $\widetilde{X}$ (Lemma 3.10), the same is true for $B_1(\sigma, X)$ in X. We conclude by the following criterion for 3-convexity (Lemma 3.6 in [JS]): a connected full and locally 3-convex subcomplex with polygonal diameter ≤ 3 in a 6-large simplicial complex X is 3-convex in X.

Definition 3.12 (combinatorial sphere). The *combinatorial sphere* $S_n(\sigma, X)$ centered at a simplex σ of a simplicial complex X is the subcomplex of the ball $B_n(\sigma, X)$ spanned by the vertices remaining at polygonal distance n (in the 1-skeleton of X) from σ.

Note that, by Lemma 3.10, balls in a systolic complex X are full subcomplexes, and hence the same holds true for spheres. In view of this, Corollary 3.5 implies the following.

Corollary 3.13. *If X is a k-systolic simplicial complex for some $k \geq 6$ then every sphere $S_n(\sigma, X)$ in X is k-large. In particular, every sphere in X is a flag complex.*

Next result may be thought of as describing certain rather strong convexity property of combinatorial balls in systolic complexes. It is a special case of Corollary 7.9 in [JS].

Lemma 3.14. *Let X be a systolic complex and τ a simplex in some sphere $S_{n+1}(\sigma, X)$. Then*

(1) *the intersection $X_\tau \cap S_n(\sigma, X)$ is nonempty, and it is a single simplex (called the projection of τ on $S_n(\sigma, X)$);*
(2) *if ρ is the projection of τ on $S_n(\sigma, X)$ then the link $[B_{n+1}(\sigma, X)]_\tau$ coincides with the ball $B_1(\rho, X_\tau)$.*

4. Osajda's inverse sequence.

In this sectiuon we recall from [O] the construction of an inverse sequence due to Damian Osajda. This sequence consists of combinatorial spheres in a 7-systolic complex, with appropriately chosen bonding maps, and its inverse limit coincides with the Gromov boundary of the complex. The maps in the sequence are simplicial (with respect to appropriate subdivision), and their closer study allows to deduce various properties of the limit.

Let X be a 7-systolic simplicial complex, and let v_0 be a fixed vertex of X. Osajda's inverse sequence consists of combinatorial spheres $S_n(v_0, X)$ centered at v_0. Construction of the bonding maps $p_n : S_{n+1}(v_0, X) \to S_n(v_0, X)$ is based on the following convexity property, which is a reformulation of Lemma 3.1 in [O]. We call the vertices in a simplicial complex *adjacent* if they are connected with an edge.

Lemma 4.1. *Let v_1 and v_2 be adjacent vertices of the sphere $S_{n+1}(v_0, X)$, and let ρ_1, ρ_2 be their projections on $S_n(v_0, X)$. Then one of the simplices ρ_i is a face of the other.*

Consider the first barycentric subdivision $[S_n(v_0, X)]'$, and for a simplex ρ of $S_n(v_0, X)$ denote by b_ρ the barycenter of ρ viewed as a vertex of $[S_n(v_0, X)]'$. The assertion of Lemma 4.1 reads then as follows: *the vertices b_{ρ_1}, b_{ρ_2} are adjacent in $[S_n(v_0, X)]'$.*

Denote by $\pi(v)$ the projection of a vertex $v \in S_{n+1}(v_0, X)$ on $S_n(v_0, X)$. By the above property and the fact that spheres are flag (because they are 7-systolic, see Coroillary 3.13), the assignment $v \to b_{\pi(v)}$ extends to a well defined simplicial map $p_n : S_{n+1}(v_0, X) \to [S_n(v_0, X)]$. The maps p_n are the bonding maps in Osajda's inverse sequence.

Damian Osajda has shown the following result (Lemma 4.1 in [O]).

Proposition 4.2. *Let X be a locally finite and finite dimensional 7-systolic simplicial complex. Then the Gromov boundary ∂X of X is homeomorphic to the inverse limit*

$$\varprojlim \left(S_n(v_0, X), p_n\right)$$

of the Osajda's inverse sequence.

5. Dual cellulations of spheres.

Let S be any simplicial complex. Define *dual cellulation* S_{dual} as follows. Let S' be the first barycentric subdivision of S, and for any simplex σ of S let b_σ be the barycenter of σ, viewed as a vertex of S'. Recall that vertices of S' are in 1–1 correspondence with simplices of S, via the map $\sigma \to b_\sigma$.

Given any simplex σ of S, denote by σ_{dual}, and call *the dual cell of* σ, the subcomplex of S' spanned by the vertex set $\{b_\tau : \tau \supset \sigma\}$. The dual cellulation S_{dual} is then the decomposition of S' formed of all dual cells $\sigma_{\text{dual}} : \sigma \subset S$. In fact, S_{dual} is isomorphic, as a stratified space, to the geometric realization of the reversed face poset of S (i.e. the poset of all nonempty faces of S with reversed order).

A *dual subcomplex* of S is any subcomplex Q of S' which is the union of certain family of dual cells.

The main result of this section, which reveals significance of the dual cellulations, is the following.

Theorem 5.1. *Let X be a 7-systolic simplicial complex, and $v_0 \in X$ a vertex. Let $p_n : S_{n+1}(v_0, X) \to [S_n(v_0, X)]'$ be the bonding map in the Osajda's inverse sequence for*

(X, v_0). *Then for any dual subcomplex Q of $S_n(v_0, X)$ the preimage $p_n^{-1}(Q)$ is a locally 3-convex subcomplex of the sphere $S_{n+1}(v_0, X)$.*

Theorem 5.1 together with Proposition 3.8 imply the next result, which will play essential role in the proof of Theorem A. Recall that a topological subspace $Y \subset Z$ is π_1-*injective* in Z if for any point $y \in Y$ the induced homomorphism $\pi_1(Y, y) \to \pi_1(Z, y)$ is injective.

Corollary 5.2. *Under assumptions of Theorem 5.1, for any dual subcomplex Q of $S_n(v_0, X)$ the preimage $p_n^{-1}(Q)$ is π_1-injective in the sphere $S_{n+1}(v_0, X)$.*

To prove Theorem 5.1, we need a serie of lemmas and some terminology. Let X be a simplicial complex, τ its any simplex, and $m \geq 1$ an integer. The *ball complement* $[B_m(\tau, X)]^c_X$ in X is the full subcomplex spanned on the set of those vertices of X which are at polygonal distance $\geq m$ from τ. Obviously, we have $B_m(\tau, X) \cap [B_m(\tau, X)]^c_X = S_m(\tau, X)$.

Lemma 5.3. *Let L be a 7-large simplicial complex and τ its simplex. Then*
(0) for any $m \geq 1$ the ball complement $[B_m(\tau, L)]^c_L$ is 7-large;
(1) the sphere $S_1(\tau, L)$ is 3-convex in the complement $[B_1(\tau, L)]^c_L$;
(2) the sphere $S_2(\tau, L)$ is locally 3-convex in the complement $[B_2(\tau, L)]^c_L$.

Proof: Part (0) follows directly from Lemma 3.3.

To prove (1), let $\gamma = (u_0, u_1, u_2)$ be a polygonal path embedded in $[B_1(\tau, L)]^c_L$, with its endpoints u_0, u_2 contained in $S_1(\tau, L)$. Suppose also that u_1 is not in $S_1(\tau, L)$. We need to show that the endpoints are at distance 1 in $[B_1(\tau, L)]^c_L$. To see this, note that we can view γ as embedded in L, with its endpoints in the ball $B_1(\tau, L)$, and with u_1 outside this ball. Now, since $B_1(\tau, L)$ is 3-convex in L (see Lemma 3.11), it follows that u_0, u_2 are at distance 1 in $B_1(\tau, L)$. However, both u_0, u_2 belong to the sphere $S_1(\tau, L)$, which is a full subcomplex of L. Thus u_0, u_2 are at distance 1 in $S_1(\tau, L)$, and hence also in $[B_1(\tau, L)]^c_L$.

To prove (2), let σ be a simplex of $S_2(\tau, L)$. We need to show that the link $[S_2(\tau, L)]_\sigma$ is 3-convex in the link $[[B_2(\tau, L)]^c_L]_\sigma$. To do this, we will apply the already proved part (1) of the lemma. For that purpose, observe that

$$[[B_2(\tau, L)]^c_L]_\sigma = (L_\sigma \setminus [B_2(\tau, L)]_\sigma) \cup [S_2(\tau, L)]_\sigma.$$

Note also that, if ρ is the projection of σ on the sphere $S_1(\tau, L)$, then $[B_2(\tau, L)]_\sigma = B_1(\rho, L_\sigma)$ and $[S_2(\tau, L)]_\sigma = S_1(\rho, L_\sigma)$. In particular, we get

$$[[B_2(\tau, L)]^c_L]_\sigma = [L_\sigma \setminus B_1(\rho, L_\sigma)] \cup S_1(\rho, L_\sigma) = [B_1(\rho, L_\sigma)]^c_{L_\sigma}.$$

Thus, we may apply (1) for L replaced with L_σ and τ replaced with ρ. This completes the proof.

Next lemma requires further terminology. Let S be a simplicial complex, and suppose its dimension is d. A *dual k-skeleton* $S^{[k]}_{\mathrm{dual}}$ is the dual subcomplex of S being the union of the dual cells σ_{dual} for all $\sigma \subset S$ with $\dim \sigma \geq d - k$.

Lemma 5.4. *Under notation of Theorem 5.1, put $S = S_n(v_0, X)$ and $d = \dim S$. Let σ be a simplex of S of dimension $d-k$ and denote by τ its projection on the sphere $S_{n-1}(v_0, X)$.*

(1) *We have*

$$p_n^{-1}(\sigma_{\text{dual}}) = [B_2(\tau, X_\sigma)]^c_{X_\sigma} \tag{5.4.1.1}$$

and

$$p_n^{-1}(\sigma_{\text{dual}}) \cap p_n^{-1}(S^{[k-1]}_{\text{dual}}) = S_2(\tau, X_\sigma). \tag{5.4.1.2}$$

(2) *Let $\bar\sigma$ be a simplex of S of dimension $d-k$, $\bar\sigma \neq \sigma$. Then*

$$p_n^{-1}(\sigma_{\text{dual}}) \cap p_n^{-1}(\bar\sigma_{\text{dual}}) \subset p_n^{-1}(S^{[k-1]}_{\text{dual}}).$$

Proof: First, observe that $p_n^{-1}(\sigma_{\text{dual}}) \subset X_\sigma$. Indeed, if v is a vertex in $p_n^{-1}(\sigma_{\text{dual}})$ then $p_n(v) = b_\pi$ for some simplex π that contains σ. Since v and π span a simplex of X, the same holds for v and σ, and thus $v \in X_\sigma$. The inclusion follows then from the fact that X_σ is full in X.

Second, note that σ_{dual} is a full subcomplex of S', and thus the preimage $p_n^{-1}(\sigma_{\text{dual}})$ is a full subcomplex in the sphere $S_{n+1}(v_0, X)$. In view of the inclusion shown in the previous paragraph, to get (5.4.1.1) it is sufficient to check that a vertex v of X_σ is in $p_n^{-1}(\sigma_{\text{dual}})$ iff $dist_{X_\sigma}(v, \tau) \geq 2$.

Since $X_\sigma \cap B_n(v_0, X) = B_1(\tau, X_\sigma)$, we immediately get one implication in the above statement. To get the other implication, suppose $dist_{X_\sigma}(v, \tau) \geq 2$. It follows that $v \notin B_n(v_0, X)$, hence $v \in S_{n+1}(v_0, X)$. Suppose that $p_n(v) = b_\pi$. Since $\pi = Res(v, X) \cap B_n(v_0, X)$, and since v and σ span a simplex of X, we get that $\sigma \subset \pi$. Thus $b_\pi \in \sigma_{\text{dual}}$, and (5.4.1.1) follows.

To prove (5.4.1.2), note that the dual skeleton $S^{[k-1]}_{\text{dual}}$ is a full subcomplex of S'. Consequently, $p_n^{-1}(\sigma_{\text{dual}}) \cap p_n^{-1}(S^{[k-1]}_{\text{dual}})$ is a full subcomplex of $p_n^{-1}(\sigma_{\text{dual}})$. Thus, we need to show that a vertex v of $p_n^{-1}(\sigma_{\text{dual}})$ is in $p_n^{-1}(\sigma_{\text{dual}}) \cap p_n^{-1}(S^{[k-1]}_{\text{dual}})$ iff $dist_{X_\sigma}(v, \tau) = 2$.

Let v be a vertex of $p_n^{-1}(\sigma_{\text{dual}}) \cap p_n^{-1}(S^{[k-1]}_{\text{dual}})$. It follows that $p_n(v) = b_\pi$ for some simplex π containing σ and of dimension $\geq d-(k-1)$. In particar, π strictly contains σ. Let u be a vertex of π not contained in σ. Since $u \in [B_n(v_0, X)]_\sigma = B_1(\tau, X_\sigma)$, we have a path of length 2 in X_σ from v to τ, through u. Thus $dist_{X_\sigma}(v, \tau) = 2$. A similar argument, which we skip, gives the converse implication, hence (5.4.1.2).

To prove (2), it is sufficient to show that $\sigma_{\text{dual}} \cap \bar\sigma_{\text{dual}} \subset S^{[k-1]}_{\text{dual}}$. We omit the straightforward argument.

Part (1) of Lemma 5.4, together with Lemma 5.3, imply the following.

Corollary 5.5. *Under notation of Lemma 5.4, the intersection $p_n^{-1}(\sigma_{\text{dual}}) \cap p_n^{-1}(S^{[k-1]}_{\text{dual}})$ is locally 3-convex in the preimage $p_n^{-1}(\sigma_{\text{dual}})$.*

Lemma 5.6. *Under notation of Theorem 5.1, put $S = S_n(v_0, X)$. Then $p_n^{-1}(S^{[k-1]}_{\text{dual}})$ is locally 3-convex in $p_n^{-1}(S^{[k]}_{\text{dual}})$.*

Proof: By Lemma 5.4(2), the complex $p_n^{-1}(S^{[k]}_{\text{dual}})$ may be viewed as obtained from the complex $p_n^{-1}(S^{[k-1]}_{\text{dual}})$ by attaching to it, independently, the complexes $p_n^{-1}(\sigma_{\text{dual}})$ for all σ in S with $\dim \sigma = d - k$. Thus, to prove the lemma, it is sufficient to prove that for each attached complex $p_n^{-1}(\sigma_{\text{dual}})$, its intersection with $p_n^{-1}(S^{[k-1]}_{\text{dual}})$ is locally 3-convex in it. Since this is exactly the assertion of Corollary 5.5, the proof is complete.

Our last preparatory result, the easy proof of which we omit, is the following.

Lemma 5.7. Under notation of Lemma 5.4, let v be a vertex of the dual cell σ_{dual} not contained in the dual skeleton $S^{[k-1]}_{\text{dual}}$. Then all vertices of $S^{[k]}_{\text{dual}}$ adjacent to v are contained in σ_{dual}.

Proof of Theorem 5.1: Put $Q^{[k]} = Q \cap S^{[k]}_{\text{dual}}$ for $0 \le k \le d = \dim S$, and call it the *k-skeleton* of the dual subcomplex Q. We will show inductively that $p_n^{-1}(Q^{[k]})$ is locally 3-convex in $p_n^{-1}(S^{[k]}_{\text{dual}})$. For $k = d$ this clearly gives the assertion.

Before getting to induction, note that all skeleta $Q^{[k]}$ and $S^{[k]}_{\text{dual}}$ are full subcomplexes of S'. Hence the same is true for their preimages through p_n as subcomplexes in $S_{n+1}(v_0, X)$. In particular, all preimages $p_n^{-1}(Q^{[k]})$ and $p_n^{-1}(S^{[k]}_{\text{dual}})$ are flag simplicial complexes (because $S_{n+1}(v_0, X)$ is flag, see Corollary 3.13).

To start induction, note that both $S^{[0]}_{\text{dual}}$ and $Q^{[0]}$ are 0-dimensional, and thus $p_n^{-1}(Q^{[0]})$ is the union of some of the connected components of $p_n^{-1}(S^{[0]}_{\text{dual}})$. In particular, it is locally 3-convex in the latter. This gives the inductive assertion for $k = 0$.

Now, suppose for some $0 \le k < d$ we know that $p_n^{-1}(Q^{[k]})$ is locally 3-convex in $p_n^{-1}(S^{[k]}_{\text{dual}})$. We need to show that $p_n^{-1}(Q^{[k+1]})$ is locally 3-convex in $p_n^{-1}(S^{[k+1]}_{\text{dual}})$.

Fix a simplex σ of $p_n^{-1}(Q^{[k+1]})$. If $p_n(\sigma)$ is not contained in $Q^{[k]}$ then Lemma 5.7 easily implies that $[p_n^{-1}(Q^{[k+1]})]_\sigma = [p_n^{-1}(S^{[k+1]}_{\text{dual}})]_\sigma$, hence local 3-convexity at σ.

It remains to deal with the case when $p_n(\sigma) \subset Q^{[k]}$. Let $(u_0.u_1, u_2)$ be a path embedded in the link $[p_n^{-1}(S^{[k+1]}_{\text{dual}})]_\sigma$, with its endpoints u_0, u_2 in $[p_n^{-1}(Q^{[k+1]})]_\sigma$, and with u_1 not in this subcomplex. We need to show that u_0, u_2 are adjacent in $[p_n^{-1}(S^{[k+1]}_{\text{dual}})]_\sigma$, or equivalently in $p_n^{-1}(S^{[k+1]}_{\text{dual}})$. We consider cases corresponding to positions of u_1.

Case 1. If $u_1 \in [p_n^{-1}(S^{[k]}_{\text{dual}})]_\sigma$, then necessarily $u_0, u_2 \in [p_n^{-1}(Q^{[k]})]_\sigma$. This follows fairly directly from Lemma 5.7. The fact that u_0, u_2 are adjacent as required follows then from local 3-convexity of $[p_n^{-1}(Q^{[k]})]_\sigma$ in $[p_n^{-1}(S^{[k]}_{\text{dual}})]_\sigma$.

Case 2. If $u_1 \notin [p_n^{-1}(S^{[k]}_{\text{dual}})]_\sigma$, we also have If $u_1 \notin p_n^{-1}(S^{[k]}_{\text{dual}})$. Let τ be the (unique) $(d-k-1)$-simplex of S with $u_1 \in p_n^{-1}(\tau_{\text{dual}})$. It follows from Lemma 5.7 that $u_0, u_2 \in p_n^{-1}(\tau_{\text{dual}})$. Since τ_{dual} is not contained in $Q^{[k+1]}$, we have

$$u_0, u_2 \in p_n^{-1}(\tau_{\text{dual}}) \cap p_n^{-1}(Q^{[k]}) \subset p_n^{-1}(\tau_{\text{dual}}) \cap p_n^{-1}(S^{[k]}_{\text{dual}}).$$

The fact that u_0, u_2 are adjacent as required follows then from Corollary 5.5.

6. Closed subsets in boundaries of 7-systolic complexes.

In this section we describe some, rather special, polyhedral expansions for arbitrary closed subsets in Gromov boundaries of 7-systolic complexes.

Let X be a locally finite infinite 7-systolic simplicial complex, and let v_0 be a vertex of X. Let $(S_n(v_0, X), p_n)$ be the inverse sequence of D. Osajda associated to (X, v_0), as described in Section 3. Since the spheres $S_n(v_0, X)$ are all nonempty and finite simplicial complexes, the inverse limit

$$\varprojlim \left(S_n(v_0, X), p_n\right)$$

is a nonempty compact Hausdorff space ([MS], Theorem 3 on p. 58). Moreover, by Proposition 4.2, this inverse limit coincides with the Gromov boundary ∂X of X, and thus it is a compact metric space.

To be consistent with the notation of Section 2, we think of the projections p_n in the Osajda's inverse sequence as of the bonding maps $q_{n+1,n}$. Accordingly, we also use the maps $q_{n,m} : S_n(v_0, X) \to S_m(v_0, X)$ for any $n > m$, and the maps $q_n : \partial X \to S_n(v_0, X)$, as defined in Section 2 (in the statement of Lemma 2.6).

Let $Y \subset \partial X$ be a nonempty closed subset, and let $z_0 \in Y$ be any point. Put $z_n := p_n(z_0)$ and recall that the inverse sequence

$$\left((S_n(v_0, X), z_n), p_n\right)$$

is a polyhedral expansion for $(\partial X, z_0)$.

We now describe a sequence of subpolyhedra $Y_n \subset S_n(v_0, X)$. For each n consider the dual cellulation of the sphere $S_n = S_n(v_0, X)$. Let W_n be the smallest dual subcomplex of S_n containing the image $q_n(Y)$. Let Q_n be the union of all dual cells of S_n that intersect W_n. Cleary, Q_n is also a dual subcomplex of S_n. Now, put $Y_{n+1} := p_n^{-1}(Q_n)$ for each $n \in \mathrm{N}$, and $Y_1 := S_1$. The main result of this section is the following.

Proposition 6.1. *Under assumptions and notation as above, the sequence of pointed polyhedra (Y_n, z_n) equipped with the restricted maps $p_n|_{Y_{n+1}}$, is a polyhedral expansion of (Y, z_0).*

Proof: We will apply Lemma 2.6 with $Z = \partial X$ and $Z_n = S_n = S_n(v_0, X)$. It is sufficient to verify conditions (1)–(3) of this lemma.

To check condition (1), recall from Section 3 the definition of combinatorial balls (or neighbourhoods) in simplicial complexes. Observe that we have $Q_n = B_2(W_n, S_n')$ for each n, where S_n' is the first barycentric subdivision of $S_n(v_0, X)$.

By defintion of p_n, the preimage $p_n^{-1}(W_n)$ is a simplicial subcomplex of S_{n+1} (for the original, not subdivided, simplicial structure on S_{n+1}). Thus the ball $B_1(p_n^{-1}(W_n), S_{n+1}')$

is a dual subcomplex of S_{n+1}. It follows that $W_{n+1} \subset B_1(p_n^{-1}(W_n), S'_{n+1})$, because clearly $q_{n+1}(Y) \subset p_n^{-1}(W_n)$. Consequently, we get

$$Q_{n+1} = B_2(W_{n+1}, S'_{n+1}) \subset B_3(p_n^{-1}(W_n), S'_{n+1}).$$

By the obvious relationships between combinatorial neighbourhoods in S_{n+1} and S'_{n+1} we also get

$$B_3(p_n^{-1}(W_n), S'_{n+1}) \subset B_2(p_n^{-1})$$

and hence

$$Q_{n+1} \subset B_2(p_n^{-1}(W_n), S_{n+1}).$$

Now, since p_n is a simplicial map from S_{n+1} to S'_n, we have

$$Q_{n+1} \subset B_2(p_n^{-1}(W_n), S_{n+1}) \subset p_n^{-1}(B_2(W_n, S'_n)) = p_n{-}1(Q_n).$$

Thus

$$p_{n+1}(Y_{n+2}) = p_{n+1}(p_{n+1}^{-1}(Q_{n+1})) = Q_{n+1} \subset p_n^{-1}(Q_n) = Y_{n+1},$$

which verifies condition (1).

Condition (2) obviously holds by the definition of Y_n.

To see that condition (3) holds true, consider on each of the simplicial complexes $S_n = S_n(v_0, X)$ the standard piecewise Euclidean metric d_n. Let U be an open set with $q_n(Y) \subset U \subset S_n$. By compactness of $q_n(Y)$, there is ϵ such that $N_\epsilon(q_n(Y), S_n) \subset U$, where

$$N_\epsilon(q_n(Y), S_n) := \{x \in S_n \mid d_n(x, q_n(Y)) \leq \epsilon\}.$$

On the other hand, by definition of Q_n, we have (for each n) that $Q_n \subset N_4(q_n(Y))$.

Since each projection p_n is a simplicial map from S_{n+1} to S'_n, there is a constant c (not depending on n), with $0 < c < 1$, such that

$$d_n(p_n(x), p_n(y)) \leq c \cdot d_{n+1}(x, y)$$

for any points $x, y \in S_{n+1}$. Consequently, for any $m > n$ and any $x, y \in S_m$ we have

$$d_n(q_{m,n}(x), q_{m,n}(y)) \leq c^{m-n} \cdot d_m(x, y).$$

It follows that

$$q_{m,n}(Q_m) \subset q_{m,n}(N_4(q_m(Y))) \subset N_{4 \cdot c^{m-n}}(q_{m,n}q_m(Y)) = N_{4 \cdot c^{m-n}}(p_n(Y)).$$

Thus, if we take m such that $4c^{m-n} < \epsilon$, we get

$$q_{m,n}(Q_m) \subset N_\epsilon(q_n(Y)) \subset U.$$

Since $q_{m,n}(Q_m) = q_{m+1,n}p_m^{-1}(Q_m) = q_{m+1,n}(Y_{m+1})$, we get $q_{m+1,n}(Y_{m+1}) \subset U$, and the proposition follows.

7. Proof of the Main Theorem

Equipped with the preparatory results of Sections 5 and 6, we are now ready to give a proof of the Main Theorem stated in the Introduction.

Let X be a locally finite 7-systolic simplicial complex. If X is finite, its Gromov boundary is empty and there is nothing to prove. If X is infinite, its Gromov boundary ∂X is a compact metric space. Let $Y \subset \partial X$ be a closed subset and let $z_0 \in Y$ be a point. We need to show that the morphism

$$i_*^{\text{pro}} : \text{pro-}\pi_1(Y, z_0) \to \text{pro-}\pi_1(\partial X, z_0)$$

induced by the inclusion $i : Y \to \partial X$ is a monomorphism in the category pro-Group.

Having chosen a vertex $v_0 \in X$, let $(S_n(v_0, X), p_n)$ be the inverse sequence of D. Osajda associated to (X, v_0), as described in Section 4. Following the notation introduced in Section 6 put $z_n = q_n(z_0)$, and consider the polyhedral expansion $((S_n(v_0, X), z_n), p_n)$ for $(\partial X, z_0)$. Consider also the polyhedral expansion $((Y_n, z_n), p_n|_{Y_{n+1}})$ for (Y, z_0), as described in Section 6.

As it was explained in Secton 2 (after Remarks 2.5), the morphism i_*^{pro} is given by the morphism (in inv-Group)

$$f = (\phi, f_n) = \big(\pi_1(Y_n, z_n), (p_n|_{Y_{n+1}})_*\big) \to \big(\pi_1(S_n(v_0, X), z_n), (p_n)_*\big)$$

such that $\phi = id_{\mathbb{N}}$ and $f_n = (i_n)_* : \pi_1(Y_n, z_n) \to \pi_1(S_n(v_0, X), z_n)$, where $i_n : Y_n \to S_n(v_0, X)$ denote the inclusions.

Now, since for each n we have $Y_{n+1} = p_n^{-1}(Q_n)$, where Q_n is a dual subcomplex in $S_n(v_0, X)$, it follows from Theorem 5.1 that the homomorphism $(i_n)_*$ is injective. By Lemma 2.3, this means that i_*^{pro} is a monomorphism, which completes the proof.

8. Comments, questions and speculations.

We discuss some context in which the property of pro-π_1-saturation may appear interesting for its further study. This context is mostly related to the problem of describing the class of all topological spaces that are (homeomorphic to) Gromov boundaries of word hyperbolic groups. Recall that if a word hyperbolic group splits over a finite or a virtually cyclic group then its Gromov boundary is some combination of the Gromov boundaries of the factors. Thus, for classification purposes it makes sense to restrict attention to groups that do not split. We will say that a group is *JSJ-indecomposable* if it does not split over a finite or a virtually cyclic subgroup.

1. Pro-π_1-saturation looks like some kind of 1-dimensionality, while hereditary asphericity as some kind of at-most-2-dimensionality. Is it true that every pro-π_1-saturated

compact metric space is hereditarily aspherical? Is it true in the class of Gromov boundaries of word hyperbolic groups? Do the two notions coincide in the latter class of spaces?

2. The 2-sphere and the 2-dimensional universal Menger space $M_{2,5}$ are not pro-π_1-saturated. Find other examples of explicit 2-dimensional spaces that are Gromov boundaries of JSJ-indecomposable word hyperbolic groups and which are not pro-π_1-saturated. Is it plausible that the class of not pro-π_1-saturated Gromov boundaries of hyperbolic groups (viewed up to homeomerphism) is very restricted and could be completely described?
3. By a result of P. Zawiślak [Z], Pontriagin sphere and Pontriagin surface Π_2 are Gromov boundaries of certain 7-systolic groups (fundamental groups of some 3-dimensional locally 7-large pseudomanifolds). By our Main Theorem, these spaces are pro-π_1-saturated. Give a direct proof of this fact. Prove (or disprove) that Pontriagin surfaces Π_p for primes $p \neq 2$, which occur as Gromov boundaries of certain right-angled Coxeter groups (see [Dr]), are pro-π_1-saturated.
4. Note that, except for the examples mentioned in the previous comment, Gromov boundaries of 7-systolic groups are not known or described explicitly. Even in dimension 2 there seem to be plenty of such spaces, and certainly there is a lot of questions. For example, are Pontriagin surfaces Π_p Gromov boundaries of 7-systolic groups? More precisely, are they Gromov boundaries of fundamental groups of some 3-dimensional locally 7-large simplicial complexes? Can we characterize or describe in some explicit way Gromov boundary of the fundamental group of an orientable 4-dimensional locally 7-large pseudomanifold? Is this space (i.e. boundary) unique?
5. Trees of manifolds (called also Jakobsche spaces $X(M^n)$, see [F] or [J]) in dimensions $n \geq 3$ are not pro-π_1-saturated. They are obtained as inverse limits of iterated connected sums of many copies of the manifold M^n, and it is not hard to realize that they contain spheres of dimension $n-1$. For $n = 3, 4$ some of these spaces are homeomorphic to Gromov boundaries of certain JSJ-indecomposable word hyperbolic groups (see Theorem 4.1 and Remark 4.2 in [PS]). Find other explicit topological spaces, especially in dimensions above 4, different from spheres and Sierpiński compacta, which are not pro-π_1-saturated and which are homeomorphic to Gromov boundaries of JSJ-indecomposable word hyperbolic groups. Note that it is not known whether the universal Menger compacta in dimensions above 4 occur as Gromov boundaries of some groups. These spaces are clearly not pro-π_1-saturated.
6. Is the class of word hyperbolic groups that are not pro-π_1-saturated closed under amalgamated free product over subgroups that are (a) infinite cyclic, (b) free, (c) surface groups, (d) any subgroups (perhaps undistorted)?
7. It is not hard to give examples of word hyperbolic systolic groups which are not 7-systolic in any obvious way. On the other hand, we have no tool to distinguish word hyperbolic groups that are systolic but not 7-systolic. Is it true that Gromov boundary of every word hyperbolic systolic group is pro-π_1-saturated?
8. Systolic groups are shown in [JS2] to have the coarsely invariant property of asymptotic hereditary asphericity. Are Gromov boundaries of all word hyperbolic asymptotically hereditarily aspherical groups pro-π_1-saturated?

9. An explicit class of groups that have pro-π_1-saturated Gromov boundary is provided by right-angled Coxeter groups with 7-large nerves. As we explain below, this class contains examples with boundaries of arbitrary finite dimension. Recall that a flag simplicial complex N determines the right-angled Coxeter group W_N given by the Coxeter system $\langle S|R\rangle$, where S coincides with the vertex set of N and $R = \{(ss')^2 : s \text{ and } s' \text{ are adjacent in } N\}$. N is then called the *nerve* of W_N.

   It was shown by Gromov that if N is 5-large (equivalently, satisfies the flag-no-square condition) then W_N is word hyperbolic (Corollary on p. 132 in [G]). In particular, this holds true if N is 7-large. Moreover, the virtual cohomological dimension $vcd(W_N)$ can be computed in terms of some homological properties of N, and if N is an n-dimensional orientable pseudomanifold, this computation gives $vcd(W_N) = n + 1$ (see [Da], Corollary 13.3.5(ii), with $T = \emptyset$ and $k = 0$). Since, by the construction of T. Januszkiewicz and the author, 7-large orientable pseudomanifolds do exist in arbitrary dimension (see Corollary 19.2 in [JS]), the class of groups that we consider in this comment contains examples with arbitrary virtual cohomological dimension. By a result of Bestvina and Mess ([BM], Corollary 1.4(e)), if Γ is a virtually torsion free word hyperbolic group then $\dim \partial\Gamma = vcd(\Gamma) - 1$. By applying this to groups W_N with 7-large nerves N, we see that their boundaries have arbitrary finite dimension.

   To see that boundaries ∂W_N of groups as above are pro-π_1-saturated, we will show that these groups are 7-systolic and then apply Main Theorem. We conclude 7-systolicity by the following (rather sketchy) line of argument. A right-angled Coxeter group W_N acts geometrically by automorphisms on some simply connected cubical complex X whose link at every vertex is isomorphic to N (see [Da], Proposition 7.3.4 together with Example 7.3.6, or [Dr], paragraph just before Proposition 3.1). We convert X into a simplicial complex by the following rather general procedure that we call *thickening*. (This procedure was first invented by T. Januszkiewicz and then independently by D. Osajda.) Replace each cubical face C of X with the abstract simplex σ_C spanned on the vertex set of C. Doing this consistently, we get the simplicial complex $Th(X)$, which is rather easily seen to have the same homotopy type as X, thus being simply connected. It is an exercise (which we leave to the reader) to show that if the vertex links of X are 7-large (which is the case when N is 7-large), then the vertex links of $Th(X)$ are also 7-large. This is sufficient for concluding that $Th(X)$ is 7-systolic. Finally, it is clear that the group W_N acts on the thickened complex $Th(X)$ geometrically, which implies that it is 7-systolic.

10. We have shown in the previous comment that Gromov boundaries of right-angled Coxeter groups with 7-large nerves are pro-π_1-saturated. On the other hand, if the nerve of a hyperbolic right-angled Coxeter group contains a full subcomplex homeomorphic to a sphere of dimension ≥ 2 then Gromov boundary of this group contains a sphere of the same dimension (corresponding to the boundary of the parabolic subgroup corresponding to vertices of the subcomplex), and thus it is not pro-π_1-saturated. Is it true that if a nerve of a hyperbolic right-angled Coxeter group contains no full subcomplex homeomorphic to a sphere of dimension ≥ 2 then Gromov boundary of this group is pro-π_1-saturated?

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