Bounded cohomology of volume preserving transformation groups

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Virtual workshop: Simplicial Volumes and Bounded Cohomology
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Overview

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Let

$$G_M = \mathrm{Homeo}_0(M, \mu),$$

the group of all compactly supported homeomorphisms of M isotopic to the identity and preserving μ.

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Goal: construct a cannonical map

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Theorem (Brandenbursky, M.)

Suppose $\pi_1(M)$ *is acylindrically hyperbolic or* $\pi_1(M) \to F_2$ *is onto. Then* $H_b^3(G_M)$ *is infinite dimensional. Moreover* $\dim \bar{H}_b^n(F_2) \leq \dim \bar{H}_b^n(G_M)$.

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Moreover $\dim \bar{H}^n_b(F_2) \leq \dim \bar{H}^n_b(G_M)$.

Theorem (Kimura)

If M *is a 2-dimensional manifold, then* $H^3_b(G_M)$ *is infinite dimensional.*

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- Gambaudo-Ghys, Polterovich: $P \colon Q(\pi_1(M)) \hookrightarrow Q(G_M)$.
- Theorems hold for certain subgroups of G_M like $\mathrm{Diff}_0(M, \mu)$, $Sympl_0(M, \omega)$.

Remarks

- Γ_b works for other (co)homologies:

$$\begin{array}{ccc}
\mathrm{H}^n(\pi_1(M)) & \xrightarrow{\Gamma} & \mathrm{H}^n(G_M) \\
\uparrow & & \uparrow \\
\mathrm{H}^n_b(\pi_1(M)) & \xrightarrow{\Gamma_b} & \mathrm{H}^n_b(G_M) \\
\uparrow & & \uparrow \\
\mathrm{EH}^n_b(\pi_1(M)) & \xrightarrow{E\Gamma} & \mathrm{EH}^n_b(G_M)
\end{array}$$

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i.e.:

- for all $f, g \in G_M$, $\gamma(f, .)\colon M \to \pi_1(M)$ is measurable
- for all $f, g \in G_M$ and $x \in M$, $\gamma(fg, x) = \gamma(f, gx)\gamma(g, x)$.

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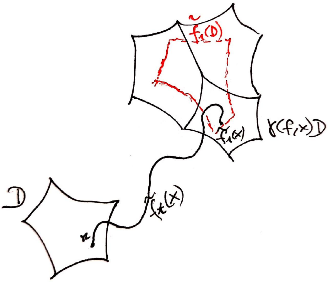

$\gamma(f,x)$ is the tile where $\widetilde{f_1}(x)$ is

i.e: $\widetilde{f_1}(x) \in \gamma(f,x)D$.

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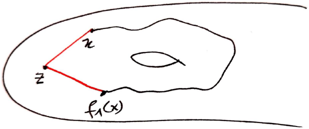

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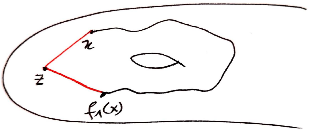

This is $\gamma(f, x) \in \pi_1(M, z)$.

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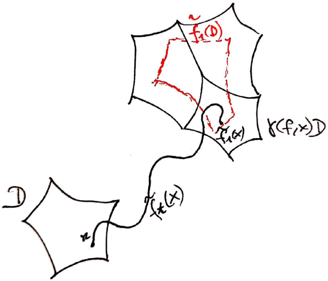
- $\gamma(f,x)$ does not depend on f_t [Ex: $ev_x\colon \pi_1(G_M) \to Z(\pi_1(M))$]
- $\{\gamma(f,x) \mid x \in M\}$ if finite.



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Notation: we always use the homogeneous resolution:
$C^n_b(G) = \{c \colon G^{n+1} \to R \mid c \text{ bd}, \ \ c(g_0 h, \dots, g_n h) = c(g_0, \dots . g_n)\}$

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- $\partial\Gamma_b = \Gamma_b\partial$
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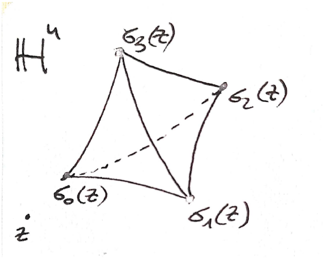
- Γ_b is cannonical (does not depend on choices)

Eaxample



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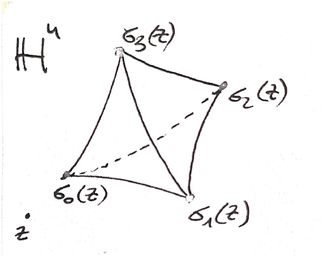

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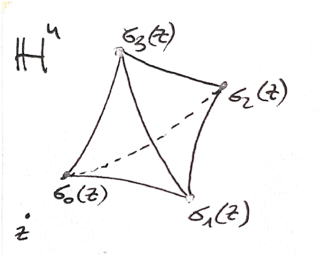

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Idea of the proof

Theorem (Brandenbursky, M.)

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Assumptions imply that there exists $\mathrm{F}_2 < \pi_1(M)$ s.t.

$$\begin{array}{c} H^n_b(\pi_1(M)) \\ \big\downarrow\!\!\downarrow {\scriptstyle i^*} \\ H^n_b(\mathrm{F}_2) \end{array}$$

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We construct $\rho_\epsilon \colon \mathrm{F}_2 \to G_M$ s.t. there exists Λ and for all $c \in H^n_b(\pi_1(M))$:

$$\|\rho_\epsilon^* \Gamma_b(c) - \Lambda i^*(c)\| \xrightarrow{\epsilon \to 0} 0.$$

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Proof:

Take $c \in H_b(\pi_1(M))$ s.t. $\|i^*(c)\| > 0$.

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Proof:
Take $c \in H_b(\pi_1(M))$ s.t. $\|i^*(c)\| > 0$.
Then $\|\rho^*_\epsilon \Gamma_b(c)\| > 0$, thus $\|\Gamma_b(c)\| > 0$.

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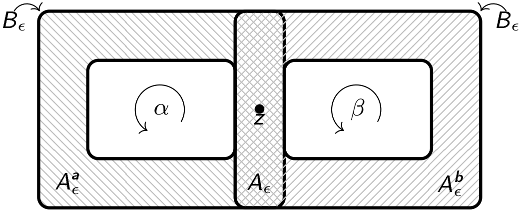

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$$i^*: H_b^n(\pi_1(M)) \twoheadrightarrow H_b^n(\mathrm{F}_2), \qquad \rho_\epsilon^*: H_b^n(G_M) \to H_b^n(\mathrm{F}_2)$$

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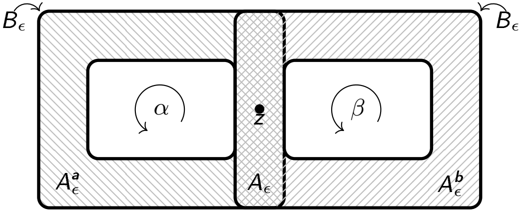

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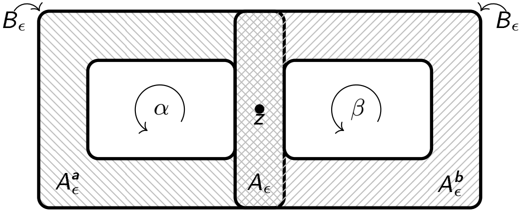

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