Examples of macroscopically large rationally inessential manifolds

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- The *macroscopic dimension* of X, denoted $dim_{mc}(X)$, is the minimal k such that there exist a k-dimensional simplicial complex K and a uniformly cobounded map $f: X \to K$.

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  E.g. if M admits a metric of non-positive sectional curvature.
- Let $f\colon M \to B\pi_1(M)$ be a map classifying the universal bundle. If $f_*([M]) = 0 \in H_n(B\pi_1(M), \mathbf{Z})$, then we can assume that the image of f is contained in $B\pi_1(M)^{[n-1]}$. Moreover, the lift of f, $\widetilde{f}\colon \widetilde{M} \to E\pi_1(M)^{[n-1]}$, is uniformly cobounded. Thus M is not macroscopically large.

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Prototypical example

Consider $M^n = N \times S^2$. Then:
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Gromov Conjecture was proven for many manifolds by Bolotov and Dranishnikov.

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Gromov-Lawson: If a spin manifold M admits a Riemannian PSC metric, then M is not enlargeable.

Definition

M is *enlargeable* if for every $\epsilon > 0$ there exist an orientable cover of M which admits an ϵ-contracting map onto S^n which is constant at the infinity and of non-zero degree.

Homological characterisation

Consider a classifying map $f\colon M \to B\pi_1(M)$. We are interested in $f_*([M]) \in H_n(B\pi_1(M), \mathbf{Q})$. If $f_*([M]) = 0$ then M is *rationally inessential*.

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Theorem (Brunnbauer-Hanke)

Let π be a finitely generated group and $n \in \mathbf{N}$. For each notion of largeness from the above list, there exist a linear subspace $H_n^{sm} < H_n(B\pi, \mathbf{Q})$ with the following property:

$$f_*([M^n]) \notin H_n^{sm} \quad \leftrightarrow \quad M \text{ is large in the respective sense.}$$

Homological characterisation

Theorem (Dranishnikov)

Assume that $B\pi$ *is compact. There exist* $H_n^{mc} < H_n(B\pi, \mathbf{Z})$ *such that:*

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Theorem (M.)

For every $n > 3$ there exist macroscopically large, rationally inessential closed smooth n-manifolds. They are not large for all large notions by the Brunnbauer-Hanke theorem.

The reflection trick of Davis

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Denote by $C(L)$ the cone of L.

The reflection trick: a recipe how to glue up some number of copies of $C(L)$ along mirrors in such a way that the resulting space, denoted by M_L, is aspherical.

The reflection trick of Davis

Special example:

We color mirrors of L on colors $e_0, \ldots, e_n$ such that non-disjoint mirrors have different colors. Assume that these colors make a linear basis of an $n+1$ dimensional vector space V over the field with two elements.

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where $\sim$ is defined as follows: assume that we are in a cone labelled by v and we cross a mirror colored by e in point x. Then we find ourself in the same point x, but in the cone labelled by $v + e$.

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- If L is a triangulation of a sphere, then M_L is a manifold.
- $\pi_1(M_L)$ is a torsion-free finite index subgroup of a right angled Coxeter group.

Outline of the construction

The construction uses the work of Davis and Januszkiewicz on small covers.

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Let L be an n-dimensional complex. Assume that $S < L$ is a subcomplex of L which is topologically an $(n-1)$-dimensional sphere. Assume moreover that $[S] \in H_{n-1}(L; \mathbf{Z})$ is a non-trivial torsion class.

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Using the reflection trick we construct an aspherical space M_L together with a subcomplex N_S given by a subcomplex S. Since S is a sphere, N_S is a manifold.

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Because of the properties of $[S]$, the class $[N_S] \in H_n(M_L; \mathbf{Z})$ is non-trivial and torsion. Moreover: $[N_S] \notin H_n^{sm}$.

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We perform a surgery on N_S to obtain a new manifold N together with a map $f : N \to M_L$ such that f is now a classifying map and $f_*([N]) = [N_S]$.

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Thus: N is macroscopically large and rationally inessential.

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In general Gromov Conjecture for N is open.

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