Exercises: Elementary Geometry
List 2. Measurable figures and their area.

1. Justify that the following figures are measurable, and that their area (i.e. their Jordan measure) is zero ($P = 0$):
   (a) a point;
   (b) a segment;
   (c) a circle of arbitrary radius R (i.e. a boundary curve only, without interior points).
2. Show that the are of any measurable figure is non-negative. Prove also that the area is monotonic: if for some two measurable figures we have the inclusin $F_1 \subset F_2$ then $P(F_1) \leq P(F_2)$.
3. Prove that if F is a measurable figure, and if F' is a figure similar to F, with scale factor k, then F' is also measurable, and we have $P(F') = k^2 \cdot P(F)$.
4. Calculate (using the same method as for the whole disk) the area of a sector of a circle of radius R having angle α at the center of the circle.
5. Show that the area (i.e. the Jordan measure) of the union of two disjoint measurable figures is equal to the sum of areas of these two figures.
6. Show that the intersection of any two measurable figures is measurable.
7. Show that for any two measurable figures F i H the following formula holds:

$$P(F \cup H) = P(F) + P(H) - P(F \cap H).$$

8. Calculate (not directly from the definition, but rather using the earlier exercises 1, 4 and 5), or derive a formula for the area of both figures obtained from a circle of radius R by cutting it along a cord of length d (so called "circular segments"). Beware that his formula may contain inverse functions for trigonometric functions.
9. Show that the following figures are measurable, and have measure zero (i.e. area equal to 0):
   (a) the union of infinitely many segments

$$\bigcup_{n=1}^{\infty} [0,1] \times \{\frac{1}{n}\};$$

   (b) the union of the boundaries of the following infinite family of the squares

$$[-\frac{1}{n}, \frac{1}{n}] \times [-\frac{1}{n}, \frac{1}{n}], \quad n = 1, 2, 3, \ldots$$

   (c) any infinite polygonal curve $A_0A_1A_2\ldots$ in which the (infinite) sum of the lengths of the constitutive segments

$$\sum_{i=0}^{\infty} |A_iA_{i+1}|$$

   is finite, i.e. the corresponding series is summable (convergent);
   (d) the set consisting of all points of the form $(\frac{1}{n}, \frac{1}{m})$, for all natural numbers n and m.
10. Verify that Sierpinski carpet is a measurable figure, and its area is 0.