Geometric Group Theory, non-positive curvature and recognition problems

Between the sea and the sky, where the weather is.

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• Lecture 1: Dehn's Problems; topological and geometric group theory — core issues; the universe of geometric group theory; Dehn functions, isoperimetric spectrum; emergence of non-positive curvature.

• Lecture 2: Non-positively curved groups; discrimination; isomorphism and homeomorphism problems; subgroup structure; subgroups of products of surface groups.

• Lecture 3: Rips constructions; balanced presentations; the resolution of Grothendieck's problems.

• Epilogue summarizing the contents of the lectures

Jim Cannon: I was greatly honoured by the invitation to speak at the festivities for Jim's 60th birthday. I have admired the clarity and depth of his work since I first encountered it as a graduate student — both his authoritative work on topological manifolds and his seminal work in geometric group theory; the latter has had a great effect on my own work. When I later learned that Jim shares my Manx roots, charm was added to beauty. *Cre t'ayd nagh vel oo gheddyn?*[1] Thank you, Jim.

Disclaimer: These notes are an edited version of notes and transparencies I prepared for my talks at Cannonfest; they are not polished in the manner of a journal article. Moreover, although I believe I have given credit to the correct individuals, I have not added detailed references. A fuller version of this material will be included in the volume based on my CBMS Lecture Series at Rensselaerville Institute, New York, August 2004.

[1]What has a man that he did not receive?

1. Combinatorial Group Theory

Presenting Groups that Arise in Nature: We consider groups given by generators and relations:

$$\Gamma = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle.$$

Why should groups be *given* in this form? I want to argue that this form emerges naturally when one confronts groups that arise in nature.

When I say "in nature", I mean that we want to understand a group Γ that arises as a group of automorphisms of some mathematical object, e.g. isometries of a metric space, linear automorphisms of a vector space, permutations of a set, diffeos of a manifold,......

In any such context, we might be interested in the group Γ obtained by composing a finite set of generating transformations of our object. In other words, we are given *generators* $\Gamma = \langle a_1, \dots, a_n \rangle$. In order to claim any understanding of the group, one needs to know which ***words*** in these generators (letters) represent the same *element* of Γ,

$$w \overset{?}{=}_\Gamma w' \ , \quad w^{-1}w' \overset{?}{=}_\Gamma 1.$$

Words equal to $1 \in \Gamma$ are called ***relations***.

Suppose we know a few relations $r_1, \dots, r_m$. How can we build on this knowledge to improve our understanding of the group? In the absence of further external knowledge, the obvious thing to do is to apply our knowledge of these basic relations repeatedly: we say the w_1 is obtained from w_2 ***by applying the relator*** r if w_2 can be obtained from w_1 by inserting r and free reduction (i.e. inserting and cancelling occurences of $a_i a_i^{-1}$).

For example, we can apply $a^{-1}b^{-1}ab$ to show that $aba = ba^2$ in $\mathbb{Z}^2$,

$$aba \overset{\text{free}}{=} ba^2.a^{-1}(a^{-1}b^{-1}ab)a \overset{\mathbb{Z}^2}{=} ba^2.$$

In general, if w' is obtained from w by applying r, then

$$w \overset{\text{free}}{=} w'(x^{-1}rx),$$

where x is a word in the generators and their inverses.

If every word w that equals $1 \in \Gamma$ can be reduced to the empty word by applying $r_1^{\pm 1}, \dots, r_m^{\pm 1}$ repeatedly — i.e. for every such w there is a free equality

$$w \overset{\text{free}}{=} \prod_{i=1}^{N} x_i^{-1} \rho_i x_i, \quad \rho_i = r_{j(i)}^{\pm 1},$$

then one writes

$$\Gamma = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle.$$

1.1. **The Basic Decision Problems.** I quote from the fundamental paper of Max Dehn, which set the direction for much of combinatorial group theory in the 20th century:

Über unendliche diskontinuierliche Gruppen (Max Dehn, 1912):

"The general discontinuous group is given by n generators and m relations between them [. . .] Here *there are above all three fundamental problems* whose solution is very difficult and which will not be possible without a penetrating study of the subject.

1. The identity [word] problem: *An element of the group is given as a product of generators. One is required to give a method whereby it may be decided in a finite number of steps whether this element is the identity or not.*

2. The transformation [conjugacy] problem: *Any two elements S and T of the group are given. A method is sought for deciding the question whether S and T can be transformed into each other, i.e. whether there is an element U of the group satisfying the relation*

$$S = UTU^{-1}.$$

3. The isomorphism problem: *Given two groups, one is to decide whether they are isomorphic or not (and further, whether a given correspondence between the generators of one group and elements of the other is an isomorphism or not).*

These three problems have very different degrees of difficulty. [. . .] *One is already led to them by necessity with work in topology.* Each knotted space curve, in order to be completely understood, demands the solution of the three above problems in a special case."

Addendum to Dehn's last comment: the context in which the basic decision problems came into being were heavily laden with **non-positive curvature** — surfaces and 3-manifolds. E.g. *every (compactified) link complement supports a metric of non-positive curvature.*

2. Topological Group Theory

In topological terms, finite presentation is a compactness criterion. There are two basic topological models for any finitely presented group.

Proposition 2.1. Γ *is finitely presented if and only if it is the fundamental group of a compact cell complex.*

Proposition 2.2. Γ *is finitely presented if and only if it is the fundamental group of a closed 4-manifold.*

The "only if" direction of Proposition 2.1 comes from the construction of the standard 2-complex of the given presentation: given a presentation $\mathcal{P} := \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$

fro Γ, one considers the combinatorial 2-complex $K(\mathcal{P})$ that has one vertex, one 1-cell for each a_i, and a 2-cell for each r_j, attached by a boundary map that traces out the combinatorial path in $K^{(1)}$ labelled by the word r_j.

One way to prove the "only if" in Proposition 2.2 is to embed the 2-complex K in $\mathbb{R}^5$ and take the boundary of a regular neighbourhood.

Anyone who feels that restricting attention to finitely presented groups is a bit wimpy can soothe their conscience with the following truly remarkable theorem of Graham Higman, which we shall discuss more in the second lecture. The hard direction of this theorem may be paraphrased as saying that every *"algorithmically describable"* group is to be found among the subgroups of finitely presented groups.

Theorem 2.3 (Higman). Γ *is recursively presented iff it is a subgroup of a finitely presented group.*

2.1. **What about better models for Γ?** The above topological models for Γ suffer from a rather gross dependence on the given presentation: even their homotopy type is not determined by Γ. To remedy this first problem, one might add cells to make $K(\mathcal{P})$ more highly connected and ultimately (but perhaps only in the limit) constructing a classifying space $K(\Gamma, 1)$. This leads us to consider

Finiteness Properties:

Definition 2.4. Γ is of type $\mathcal{F}_n$ if there is a $K(\Gamma, 1)$ with only finitely many cells in the n-skeleton.

There is a rich field of mathematics that studies such finiteness properties, with early contributions by Wall and Serre. The following important early theorem is due to Bieri, who was building on earlier work of Stallings.

Theorem 2.5. *For all $n \in \mathbb{N}$, there exist groups that are of type $\mathcal{F}_{n-1}$ but not $\mathcal{F}_n$.*

We shall see explicit examples later.

Manifold Models:

It is natural to ask what extra structure one can impose the manifold in Proposition 2.2. Can one lower the dimension, or require it to support various geometric structures? The pusuit of such questions can prove beneficial in two ways: if all groups are fundamental groups of manifold with additional structure, then one might hope to use that structure to prove interesting facts about arbitrary finitely presented groups[2]; on the other hand, if by imposing restrictions on the type of manifolds considered one finds that one excludes certain fundamental groups, then one has identified a distinguished class of groups that is likely to be worthy of further study.

[2] to the dubious extent that there exist such facts

In this vein, it is worth noting the following amalgamation of theorems of Gompf, Taubes and others:

Theorem 2.6. Every *finitely presented group is the fundamental group of a closed* symplectic *4-manifold, and of a closed* complex *manifold (of real dimension 6). But* not *every finitely presented group is the fundamental group of a* 3-manifold*, nor of a* Kähler *manifold.*

Basic Issues: Let's note some of the basic questions that arise in this context.

• Given a finitely presented group, ask what are the optimal topological/geometric (cell-complex or manifold) models for the group.

• Uniqueness issues: does imposing a suitable condition lead to a unique model (e.g. Borel conjecture — if there at most one compact aspherical manifold with given Γ as fundamental group).

• What is the relationship between different notions of dimension arising from different models — e.g. action dimension (à la Bestvina *et al.*), geometric and cohomological dimension (Serre, Wall etc.), CAT(0) dimension (Bridson and Brady-Crisp) *etc.*.

Special Classes Arising: *3-manifold groups, Kähler groups, PD(n) groups, groups with balanced presentations, 1-relator groups, Thompson-type groups,...*

3. Geometric Group Theory

We already strayed into geometric group theory in the previous section. This vibrant subject, which formed a separate identity sometime in the late 1980s, has two main strands:

Strand 1:

[... *cont. from above*] Study (and manufacture) group actions (preferably by isometries) on spaces in order to elucidate the structure of both the groups and the spaces.

Example 3.1. *Here is an example of a classical problem in combinatorial group theory that was solved only be construction of an appropriate group action on any appropriate metric space: the* conjugacy problem for Coxeter groups. *Recall that a Coxeter group (or generalized reflection group) is given by a presentation*

$$\Gamma = \langle s_1, \dots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = 1 \rangle,$$

where the m_{ij} are positive integers of ∞(= no relation, by convention).

• *(Gabor Moussong) Γ acts properly and cocompactly by isometries on a CAT(0) polyhedral complex.*

• *(Alonso–Bridson) Groups that admit such actions have a solvable conjugacy problem.*

Hence

Theorem 3.2. *All Coxeter groups have a solvable conjugacy problem.*

The preceding discussion exemplifies the idea that one can elucidate deep algebraic structure by studying the geometry of a group acting on a suitable space. In the opposite direction — using group theory to elucidate the structure of spaces — one again finds a prime exemplar in the context of Coxeter groups: Mike Davis gave a beautiful construction of a compact aspherical manifold not covered by Euclidean space by using Coxeter groups in an ingeneous manner in a context where one again finds non-positive curvature playing a crucial role. It is natural to view his construction in the broader context of complexes of groups.

3.1. **Complexes of Group.** [André Haefliger and...] In the spirit of Strand 1, you are interested to know if a certain type of group action exists. For example, suppose that whilst dreaming of wonderful 2-complexes with actions that solve all manner of problems for you, you meet a man in street who offers to sell you the action of your dreams: a group acting cellularly on a connected 2-complex, with fundamental domain a single (square) 2-cell, and cell stabilizers as shown below (where $B(x,y) = \langle x, y \mid x^{-1}yx = y^2\rangle$):

$$\begin{array}{ccccc} B(a,b) & = & B(a,b) & = & B(a,b) \\ \uparrow & & \uparrow & & \uparrow \\ \langle a\rangle & \leftarrow & \{1\} & \rightarrow & \langle b\rangle \\ \downarrow & & \downarrow & & \downarrow \\ B(a,c) & \leftarrow & \langle c\rangle & \rightarrow & B(b,c) \end{array}$$

Should you purchase this wonderful action from him?

No! For such an action does not exist. One knows this because the push-out (colimit) of the above diagram in the category of groups is the trivial group.

Explaining why this point is crucial, and initiating the search for conditions under which one can construct actions from such putative quotients, marks the beginning of the theory of complexes of groups. Early input into this theory came from Thurston (in the case of orbifolds) and more particularly Gromov, also Bass and Serre (actions on graphs), and Stallings–Gersten (triangles of groups). But the full theory was developed by André Haefliger, and is explained in considerable detail in our book [Bridson–Haefliger].

The most useful theorem concerning *developability* (the issue of when a putative quotient is realised by an actual group action) states that under mild regularity conditions, an obvious local model for the action should support a metric of non-positive curvature. Thus we see another important instance (like the conjugacy problem for Coxeter groups), where non-positive curvature makes an unexpected entrance to solve a natural problem in group theory.

Strand 2: Groups as Metric Spaces (Cayley, Dehn, Cannon, **Gromov**):

Regard finitely generated groups as metric objects and study their large-scale geometry.

Fix a group with a finite generating set:

$$\Gamma = \langle a_1, \dots, a_n \rangle, \qquad \mathcal{A} := \{a_1, \dots, a_n\}.$$

Associated to $\mathcal{A}$ one has the **word metric** on Γ:

$$d(\gamma_1, \gamma_2) = \min\{|w| : w \in F(\mathcal{A}),\ w \overset{\Gamma}{=} \gamma_1^{-1}\gamma_2\}.$$

This is the restriction to Γ (= vertex set) of the path metric on the associated **Cayley Graph (Arthur Cayley 1878, Max Dehn 1905)** in which each edge has length 1.

Note that if Γ is finitely presented, say $\Gamma = \langle \mathcal{A} \mid \mathcal{R} \rangle$, then this Cayley graph is just the 1-skeleton of the universal cover of the standard 2-complex $K(\mathcal{P})$.

Figures: at this point in the lectures I show transparencies with pictures of the Cayley graphs of $\mathbb{Z}^2$ (the rectangular lattice in $\mathbb{E}^2$) of F_2 (a regular 4-valent tree), and the 2-3-7 triangle group, drawn in the hyperbolic plane.

Then, I draw the Cayley graphs of $\mathbb{Z} = \langle t \rangle$ with respect to the generating sets $\{t\}$ and $\{t^2, t^3\}$: this illustrates the fact that the Cayley graphs of a fixed group with respect to different finite generating sets may be non-homeomorphic; but one also sees that there is something essentially linear about the large-scale geometry of the both graphs and this leads us to:

Lemma 3.3. *The Cayley graphs associated to different choices of finite generating sets for Γ are* **quasi-isometric**.

Recall

Definition 3.4. A map $f : X_1 \to X_2$ between metric spaces is a *quasi-isometry* if there exist constants $\lambda \geq 1$, $\varepsilon \geq 0$, $C \geq 0$ such that

$$\frac{1}{\lambda} d(x,y) - \varepsilon \leq d(f(x), f(y)) \leq \lambda d(x,y) + \varepsilon$$

for all $x, y \in X_1$, and every point of X_2 is within a distance C of the image of f.

The following celebrated lemma provides a bridge between the first and second strands of geometric group theory.

Proposition 3.5 (Milnor-Švarc). *If Γ acts properly and cocompactly by isometries on a geodesic space X, then Γ is quasi-isometric to X.*

And the following landmark theorem of Misha Gromov opened the mind of the world to the fact that a remarkable amount of subtle algebraic information is encoded into the seemingly crude asymptotic (q.i. invariant) geometry of a finitely generated group.

One says that a finitely generated group Γ with word metric d has polynomial growth if there exists an integer m and a constant k such that $|\{\gamma : d(1, \gamma) \leq n\}| \leq kn^m$ for all $n \in \mathbb{N}$.

Theorem 3.6 (Gromov)**.** Γ *has* polynomial growth if and only if Γ contains a *nilpotent subgroup of finite index.*

Having polynomial growth is easily seen to be an invariant of quasi-isometry, hence:

Corollary 3.7. *Having a* **nilpotent** *subgroup of finite index is an invariant of* **q.i.**.

In the wake of this theorem, there have been a string of beautiful **q.i-rigidity** results, many generalising classical rigidity results à la Mostow, but others exhibiting rigidity in utterly new settings (e.g. Baumslag-Solitar groups). In such theorems, various classes of groups (e.g. lattices in a fixed semi-simple Lie group) are shown to be virtually closed under quasi-isometry: if Γ_1 has a subgroup of finite index that is in the class and Γ_2 is quasi-isometric to Γ_1, then there is a group Γ in the class and a short exact sequence of groups $1 \to N \to \Gamma_2^0 \to \Gamma \to 1$ with N is finite and $\Gamma_2^0 \subset \Gamma_2$ of finite index. In many settings (but by no means all) there are companion theorems to the effect that groups in the class are quasi-isometric if and only they have isomorphic subgroups of finite index.

4. The Universe of Geometric Group Theory

Let me now try to sketch the universe of geometric group theory. This sketch is an honest attempt to sketch universal ideas, but it inevitably reflects my own tastes and experiences in its scale and emphases. I encourage readers to draw their own universe of groups, trying to locate on the map whatever classes of groups matter most to them, and challenging themselves to explore how the borders of their class cross the borders of other classes of groups. A particularly profitable exercise, in my experience, is to ask what happens to your universe when you hit it with various operations, e.g. the taking of finitely generated subgroups (see lecture 2).

[In the lecture, I sketch at the board as I tell the following story.]

There in an obvious first group, namely the trivial group. The finite groups might reasonably be said to come next, indeed if one takes the view that quasi-isometric groups should be identified then finite groups are indistinguishable from the trivial group. More technically, they are *virtually trivial*[3], i.e. contain the trivial group as a subgroup of finite index. For this reason we think of finite groups as constituting a (fat and interesting) point at the beginning of the universe.

What comes next? The simplest infinite group is surely $\mathbb{Z}$, so we have a second fat point representing the virtually cyclic groups.

4.1. The amenable versus the hyperbolic.

Now we come to a genuine divergence of opinion: historically, we teach our undergraduates that the next-simplest group is $\mathbb{Z}^2$, or more generally finitely generated abelian groups. Whether it is just historical accident and social convention, or whether humanity has a more innate longing for the commutative,

[3]I do not wish for a moment to demean finite group theory by this remark; I merely wish to point out immediately that finite groups play only a very minor role in the view of the universe I am articulating

I do not know, but the fact is that we all seem to be brought up thinking that the easiest way to combine two groups is to take their direct product, and hence we pass from $\mathbb{Z}$ to $\mathbb{Z}^n$. On the other hand, if you give us two lumpy spaces and ask us to describe the easiest way to combine them into a connected space, we might well stick them together at a single point: a much freer operation that (via the Seifert-van Kampen theorem) corresponds to taking the free product of groups.

Thus, immediately beyond the canonical departure point of virtually cyclic groups, the universe of groups divides into the virtually abelian world of the commutative thinkers and the world of *free-thinkers*; in the latter approach the most natural generalisation of $\mathbb{Z}$ is a finitely generated free group.

At this point, I draw a 2-dimensional sketch of the universe of finitely presented groups that is a cone flaring from left to right, with two fat points at the origin (virtually trivial and virtually cyclic groups), with the top side of the cone labelled the *amenable side of the universe* and the bottom side labelled the *hyperbolic side of the universe.*

Along the amenable side (which I habitually draw at the top for no good reason) we have the following increasing sequence of classes of groups: v.abelian, v.nilpotent, v.solvable, elementary amenable, and amenable.

Along the hyperbolic (bottom) side of the universe we have v.free groups, followed by an unlabelled region that I'll come back to, included in the larger region of hyperbolic groups, which is included in a large region labelled non-positively curved groups (NPC) that has a well-defined border near the beginning of the universe that includes v.abelian groups but excludes all other groups on the amenable side of the universe. As one extends further to the right, the borderline of NPC becomes broken and peters out. If one wanted to introduce colour, then the NPC colour should fade as one moves to the right, indicating that as one enlarges the class of groups, what remains of the essence of NPC weakens. *This is a theme that will form the focus of much of what follows, as will with task of dividing this region NPC into smaller regions corresponding to classes of groups that each have a reasonable claim to be called non-positively curved.*

4.2. **Here there be dragons!** You may notice that many famous conjectures about arbitrary groups are proved for the groups along the top or the bottom of the universe. One should be *very* careful of thinking that such results are evidence for the truth of a conjecture about arbitrary groups. In the vast region between the amenable side of the universe and the hyperbolic side there lurk all manner of monsters. It is here that one finds the groups that show almost all algorithmic problems about arbitrary finitely presented groups are undecidable. The groups that occupy this jungle have little or nothing in common with the groups on the two well-behaved coasts of the universe. Thus extrapolating evidence from the coasts to conjectures about the interior is extremely hazardous.

4.3. **Taking limits.** I will not say much more about the top (amenable side of the universe), but let me point out that in the light of Gromov's polynomial growth theorem, v.nilpotent groups have an indisputable claim to the ground next to abelian groups: they are exactly the groups of polynomial growth; they also form the largest class of groups Γ such that the sequence of rescaled Cayley graphs $X_n = (\Gamma, \frac{1}{n}d)$ converge in the pointed Gromov-Hausdorff topology (as proved by Pansu).

Whilst thinking in terms of taking limits, let us ask what class of groups one gets if one takes limits of free groups. More precisely, suppose that one fixes a finitely generated free group F_r and asks which groups Γ have Cayley graphs that are the pointed Gromov-Hausdorff limit of a sequence of Cayley graphs for (F_r, S_n), where S_n is a varying choice of finite generating set (of fixed cardinality). The answer is *limit groups*. These are a particular class of groups, non-positively curved in every reasonable sense, that have emerged in the last couple of years as a class of fundamental importance, thanks largely to the groundbreaking work of Zlil Sela.

One measure of the importance of limit groups is the diversity of senses in which they claim the ground immediately next to free groups in our universe. We have made one claim in the language of GH-limits; a quite different claim can be made from the point of view of logic. Given any finitely generated group one can consider the universal theory of that group, i.e. the collection of all sentences in first order logic, which involve only the quantifier $\forall$ (together with group elements and the group operations, $\vee$, $\wedge$ and **not**, but not any mention of subsets). Limit groups are precisely the groups which have the same universal theory as a free group. More fundamentally, Sela characterises the groups that have the same *elementary theory* as a free group (first order sentences with the quantifiers $\exists$ and $\forall$) as a particular subclass of the hyperbolic limit groups.

Let's return to the idea of trying to take GH-limits of $X_n = (\Gamma, \frac{1}{n}d)$, where Γ is a finitely generated group with fixed word metric d. As we've noted, the limit will not exist if Γ is not v.nilpotent. However, one can circumvent this difficulty by fixing a non-principal ultrafilter and taking an *ultralimit*. If one does this for any finitely generated free group, then the ultralimit will be an everywhere-branching $\mathbb{R}$-tree. Thus it seems reasonable to say that adjoining the class of free groups one might take the class of finitely generated groups that also have this property. This turns out to be the class of groups that are *hyperbolic in the sense of Gromov*. This is just one of the many avenues by which hyperbolic groups appear as the most commanding generalisation of a free group; I shall briefly recall some of the other compelling evidence.

4.4. **Hyperbolic Groups (à la Gromov).**

Definition 4.1. Fix $\delta > 0$. A geodesic metric space X is δ-hyperbolic if each side of each geodesic triangle in X is contained in the δ-neighbourhood of the union of the two sides. A finitely generated group is δ-hyperbolic if its Cayley graph w.r.t. any finite generating set is δ-hyperbolic.

The Cayley graph of a group Γ with respect to a finite generating set S is a tree iff Γ is a free group with basis S. The constant δ is a measure of how far a Cayley graph is from being a tree, and a group is hyperbolic iff this measure is finite. Since geodesic triangles in in the hyperbolic plane $\mathbb{H}^2$ have uniformly bounded area, there is a bound on the size of hemidiscs that one can embed in them, and this provides a δ for the hyperbolic plane. Standard comparison theorems in Riemannian geometry show that small geodesic triangles in manifolds of sectional curvature ≤ -1 are no fatter than triangles in $\mathbb{H}^2$ with the same side lengths, so the same δ works for the universal covering of such manifolds.

A key feature of hyperbolic spaces is that quasi-geodesics stay uniformly close to geodesics, and it follows easily from this that hyperbolicity (but not the optimal value of δ) is an invariant of quasi-isometry. In the light of the Milnor-Švarc lemma, it follows that non-Euclidean surface groups and fundamental groups of closed negatively curved manifolds (more generally oribfolds) are δ-hyperbolic. A central theme in the study of these groups is the fact that arbitrary hyperbolic groups share many of the features of such fundamental groups. For example:

Theorem 4.2. *If Γ is δ-hyperbolic then*

(1) Γ is finitely presented;
(2) Γ acts on a contractible complex with compact quotient and finite stabilizers;
(3) Γ has only finitely many conjugacy classes of finite subgroups;
(4) all abelain subgroups of Γ are virtually cyclic.

Much has been written about hyperbolic groups and I'll assume that you've heard or read at least part of it. But, in the spirit of these lectures, let me emphasize a couple of points: the first is that hyperbolic groups deserve their substantial place in our universe by virtue of their interesting properties, their ubiquitousness (in a certain precise statistical sense, most finite presentations describe hyperbolic groups), and the diversity of their claim. Secondly, an important feature of hyperbolic groups is that they are the groups with the most tractable word problems possible in a sense that we shall make precise in the following section. From the point of view on group theory that I have emphasized this is particularly important: I immediately stressed the central role of the word problem and now I am trying to sketch a view of the universe which begins with the easiest types of groups (finite, cyclic, free, abelian,...) and builds outwards. In such a scheme, a special place must go to the class for which the most natural measure of complexity for the word problem (the Dehn function) is as small as possible, and these are the *hyperbolic groups.*

We shall see later that although they can be defined purely in terms of the complexity of their word problems, hyperbolic groups also have well-behaved conjugacy and isomorphism problems.

5. Dehn functions and isoperimetric behaviour

Recall from our initial discussion of group presentations that

$$\Gamma = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$$

means that a word w in the free group on the a_i equals $1 \in \Gamma$ if and only if there is an equality in the free group

$$w \overset{\text{free}}{=} \prod_{i=1}^{N} x_i^{-1} \rho_i x_i, \quad \rho_i = r_{j(i)}^{\pm 1}.$$

The **word problem** challenges us to decide, given any word w, whether such an equality exists. The difficulty of solving this problem directly is obviously tied-up with the size of the integer N in a smallest such equality, which is called the **area of** w. To explore this idea geometrically, recall the correspondence

$$\mathcal{P} \cong \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle \leftrightarrow \text{2-complex } K(\mathcal{P})$$

This induces a correspondence,

$$\text{words in } a_1^{\pm 1}, \dots, a_n^{\pm 1} \;\leftrightarrow\; \text{loops in } K^{(1)}.$$

Words representing the identity in Γ correspond to null-homotopic loops in $K^{(1)}$. Indeed there is a correspondence between free-equalities and combinatorial homotopies

$$w \overset{\text{free}}{=} \prod_{i=1}^{N} x_i^{-1} r_i x_i \;\leftrightarrow\; \text{homotopy diagrams.}$$

Such homotopy diagrams — more commonly, **van Kampen diagrams** — are planar 1-connected 2-complex that admit a combinatorial map to K so that the boundary is mapped to the edge-loop labelled w. *Van Kampen's Lemma* states that the **area** of w is the least number of faces in any van Kampen diagram for w. (A careful proof of this result is given in my essay *"The Geometry of the Word Problem"*[4].) [draw pictures]

Definition 5.1. The *Dehn function* of a finitely presented group $\Gamma = \langle A \mid R \rangle$ is the least function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \geq \text{Area}(w)$ for all words $w \in F(A)$ of length $\leq n$ such that $w = 1$ in Γ.

Using van Kampen diagrams, one shows that the Dehn function of a group is $\simeq$invariant up to quasi-isometry; in particular its asymptotic behaviour depends only on Γ and not on any particular presentation of it. Thus one may speak of a group as having a Dehn function that is polynomial of a given degree (or exponential, or $\simeq n^{5/2}$ or $\simeq n \log n$, etc.). The minimal $\simeq$equivalence class is comprised of the linear functions.

The following fundamental theorem builds on work of Max Dehn and Jim Cannon, and is due to Mishal Gromov (with various detailed proofs by different authors). **It shows that negative curvature (hyperbolicity) is not just an artefact of a particular attempt to understand groups but rather it emerges perforce at the first step in a direct analysis of the word problem.**

Theorem 5.2. *A finitely presented group* Γ *has a linear Dehn function if and only if* Γ *is hyperbolic. Moreover, if the Dehn function of a group is* $0(n^2)$*, then it is linear.*

[4]"Invitations to Geometry and Topology", Oxford Univ. Press, 2002.

The second phrase of the theorem implies that the class of groups with quadratic Dehn functions is also worthy of particular attention. We shall return to this point in the next lecture, but let me add weight to this point here by pointing out that Noel Brady and I proved that the gap between linear and quadratic is **the only gap in the isoperimetric spectrum**, i.e. the set $\mathrm{IP} = \{\alpha \mid n^\alpha \simeq \text{a Dehn function}\}$ is dense in $\{1\} \cup [2, \infty)$.

Finer information about the structure of Dehn functions was obtained in the annals paper *Isoperimetric and isodiametric functions of groups* by Sapir, Birget and Rips.

Strategy

The strategy underpinning these lectures (and much of my life) is this:

Stand in the region of the universe of finitely presented groups labelled NPC, with hyperbolicity at your back and the wild universe of arbitrary groups in front: "between the sea and the sky, where the weather is".

Encode fundamental problems from elsewhere in mathematics into group presentations, accepting that the groups you get are likely to be wild. Then develop techniques that enable you to reach out into the wild universe and translate the phenomena you're trying to understand into the world of non-positively curved groups (typically, into the subgroup structure of such groups, or bad sequences of nice groups).

The Higman embedding is a prime tool for achieving the first stage of the above strategy. We shall discuss techniques for achieving this second stage in later Sections.

Prototypes of what one hopes to achieve:

Conjugacy Problem for Coxeter Groups (as discussed earlier).

Mike Davis's construction of aspherical manifolds not covered by $\mathbb{R}^n$ (using reflection group techniques and ideas closely related to complexes of groups).

5.1. **A vindication of this way of life.** In the final lecture of this series I'll describe recent[5] work by myself and Fritz Grunewald in which (staring out into the universe of groups from the edge of NPC) we solve two problems posed by Grothendieck in 1970. The least technical of these problems is the following.

Given a commutative ring $A \neq 0$ and a group Γ, let $\mathrm{Rep}_A(\Gamma)$ denote the category of Γ-actions on finitely presented A-modules. Any homomorphism of groups $u : \Gamma_1 \to \Gamma_2$ induces a functor $u_A^* : \mathrm{Rep}_A(\Gamma_2) \to \mathrm{Rep}_A(\Gamma_1)$ by restriction of scalars.

Question (Grothendieck, 1970): *Let* Γ_1 *and* Γ_2 *be finitely presented, residually finite groups and let* $u : \Gamma_1 \to \Gamma_2$ *be a homomorphism. If* $u_A^* : \mathrm{Rep}_A(\Gamma_2) \to \mathrm{Rep}_A(\Gamma_1)$ *is an equivalence of categories for every* $A \neq 0$*, does it follow that* u *is an isomorphism?*

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[5]at the time of the lectures; it appeared in the Annals of Math, 2004

Cannonfest Lecture 2

6. What is a non-positively curved group?

Well, there are certain classes that must certainly be included:

• Hyperbolic groups (à la Gromov) must be NPC;

• likewise groups that act properly and cocompactly by isometries on CAT(0) spaces;

• semihyperbolic groups [Bridson-Alonso 1992];

• automatic groups [Cannon, Epstein, Holt, Levi, Patterson and Thurston];

• combable groups [the boundary of NPC?]

• IP(2) ?!? This is the class of groups whose Dehn function is either linear or quadratic.

• Subgroups?!? Does the theory of NPC groups change radically if we insist on subgroup closure properties? (Answer: Yes, as we shall see.)

6.1. **Definitions.** Gromov's highly successful theory of hyperbolic groups begins with the identification of a property that encapsulates the essence of the geometry of geodesics in negatively curved manifolds: the thin triangles condition[6]. Note that the encapsulation has to be robust enough to be inherited by the Cayley graph of the fundamental group Γ of a compact manifold M via the quasi-isometry $\gamma \mapsto \gamma \cdot x$ induced by a choice of basepoint $x \in \tilde{M}$.

If one takes the same approach to non-positively curved spaces, then one is immediately led to approximate geodesics in $\tilde{M}$ by suitable paths in the Cayley graph of Γ. The most straightforward way of doing this leads to the notion of a **combable** group:

Combable groups: If Γ is a group with finite generating set S then a *combing* is a section $\sigma : \Gamma \to \Sigma^*$ of the natural map $\text{Free}(S) \to \Gamma$.

One regards $\sigma(g)$ as a *discrete path* from $1 \in \Gamma$ to g: at integer times the path visits the image in G of the prefix of $\sigma(g)$ that has length t.

The combing is said to satisfy the *fellowtraveller* property if paths to adjacent vertices stay uniformly close: $\exists K > 0$ s.t. $\forall g, g' \in \Gamma$,

$$D(\sigma(g), \sigma(g')) := d(\sigma(g)_t, \sigma(g')_t) \leq K\, d(g, g')$$

for all $t \leq \max\{|\sigma(g)|, |\sigma(g')|\}$.

If Γ has such a combing, it is said to be *combable.* If, in addition, $D(s.\sigma(a^{-1}g), \sigma(g')) \leq K\, d(g, g')$ for all $s \in S$, then Γ is said to be *bicombable.*

Semihyperbolic groups: One does not get any *a priori* bound on the length of $\sigma(g)$ in terms of $d(1, g)$ from the fellowtraveller property. But the (bi)combings that fundamental

[6] which has various equivalent guises

groups inherit from the geometry of the universal cover are quasi-geodesics. In 1992 Juan Alonso and I developed a theory of *semihyperbolic groups*, taking as our definition that the group should be bicombable with a combing by quasi-geodesics. It transpires that in this context one can recover almost all of the rich structure of fundamental groups of compact non-positively curved (i.e. locally CAT(0)) spaces.

6.2. **Automatic groups.** I'll dwell on this because it was so important in the development of modern geometric group theory. We have discussed geometric constraints on combings, but there is something more subtle type of constraint that begins with a remarkable insight of Jim Cannon, namely *lingusitic simplicity.*

The *cone type* of $\gamma \in \Gamma$ is the view from γ looking away from the identity in the Cayley graph, that is

$$C(\gamma) = \{w \in (S^{\pm 1})^* \mid d(1, \gamma w) = d(1, \gamma) + |w|\}.$$

Theorem 6.1 (Cannon). *If* Γ *is* hyperbolic *then* Γ *has only finitely many cone types.*

Corollary 6.2. *The language of geodesics in a hyperbolic group is* regular*, i.e. there is a finite state automaton that can recognise whether or not a word in the generators is a geodesic.*

[Draw an automaton that recognises geodesics in the free group of rank 2]

Motivated by this, one defines a group to be *automatic* if it is combable and the language of the combing $\{\sigma(g) \mid g \in \Gamma\}$, viewed as a subset of the free monoid $(S^{\pm}1)^*$ is a *regular language.*

This turns out to be both a practically useful and strikingly subtle condition. On the practical side, it enables one to do genuine experiments in groups, finding automatic structures and drawing accurate pictures of large parts of their Cayley graphs, for example. On the theoretical side, one can prove many attractive theorems about automatic groups. Moreover, the theory is animated by large classes of examples that are of intrinsic interest, for example 3-manifold groups (excluding those that have a connected summand that fibres over the circle), mapping class groups [Mosher], and braid groups. The theory of automatic groups began with conversations between Cannon and Thurston following Cannon's seminal paper on the algorithmic structure of Kleinian groups, and blossomed in a project also involving Epstein, Holt, Levy and Patterson, with Epstein directing the writing of their important monograph on the subject.

There is a corresponding notion of biautomatic group based on bicombings, introduced by Gersten and Short. The additional structure in this setting allows one to prove stronger results about the conjugacy problem and commutativity, but at the time of writing there are still no example known of an automatic group that is not biautomatic.

6.3. **Quadratic isoperimetric inequalities.**

Theorem 6.3. *If* Γ *is the fundamental group of a compact non-positively curved space, or is semihyperbolic, or is automatic, then* $\Gamma \in IP(2)$*, i.e. the Dehn function of* Γ *is either linear or quadratic.*

In the first two cases this is straightforward once one knows how to construct van Kampen diagrams from constraints on combings: it a coarse analogue of the fact that if one has a loop of length l in a CAT(0) space (e.g. Euclidean space) then one can fill it with a disc of area at most Kl^2 simply by coning it off to any point p on the perimeter using the unique geodesics joining p to the other points of the loop; in the setting of combable groups, one imitates this with combing lines in place of geodesics; provided the combing lines are quasi-geodesic, one still gets a quadratic bound.

This raises two questions: first, are all the groups in IP(2) deserving of the title "non-positively curved group"; secondly, do all combable groups lie in IP(2)?

I shall not go into the reasons why, as it would take us too far afield, but the fact is that the membership of IP(2) is very diverse and some of its members do not belong in the pantheon of non-positive curvature: in terms of our map of the universe, IP(2) cuts a considerable swathe across towards the amenable side, embracing many nilpotent groups that are not virtually abelian, for example.

6.4. **Discrimination.** The second of the above questions remained unanswered for ten years from the late eighties, along with the questions: *Is every combable group automatic? Is every combable group bicombable?*

I answered them in a paper that appeared in Comm. Math. Helv in 2002.

Theorem 6.4. *There exist combable groups that are not bicombable.*

Theorem 6.5. *There exist combable groups whose Dehn functions are cubic.*

Corollary 6.6. *There exist combable groups that are not automatic.*

Further distinctions, e.g. bicombable but not automatic; IP(2) combable but not automatic are made in a subsequent set of notes, *"Combing through the remnants of non-positive curvature"*.

7. Decision Problems

I began these lectures by emphasizing the central role that the basic decision problems play in the study of finitely presented groups. In this section I'll explain why the issue of "undecidability" is much more mundane and concrete than one might imagine; in particular I'll explain why the basic decision problems (and all reasonable variations on them) are unsolvable in the absence of constraints on the finitely presented groups considered. We shall then see how undecidability phenomena for groups combine with the construction of manifolds with arbitrary fundamental group to show that recognition problems for closed manifolds are also algorithmically unsolvable in general.

7.1. **What is an unsolvable problem?** Fix a finite set A. A set of words $S \subset A^*$ is *r.e. (recursively enumerable)* if there exists a Turing machine that can generate a complete and accurate list of the elements of S. The set S is *recursive* if both S and $A^* \smallsetminus S$ are r.e.

At the heart of undecidability results is:

Theorem 7.1. *There exist recursively enumerable subsets (of* $\mathbb{N}$*) that are not recursive.*

Equivalently:*there are* r.e. *sets for which the membership problem is unsolvable.* This is proved by a simple diagonalization argument that one can give in an undergraduate course; it is equivalent to the undecidability of the halting problem.

7.2. **Translation Into Group Theory.** Nothing is hidden in what follows. My purpose is to demonstrate how the existence of r.e. sets that are not recursive leads *easily* to a proof the all of Dehn's basic problems are undecidable in general, *provided* one accepts Higman's Embedding Theorem. Without this theorem, all of the results beyond Propostion 7.2 are very hard to prove. Thus my treatment is designed to point out what a powerful theorem Higman has provided us with.

First the word problem:

Proposition 7.2. *If* $S \subset \mathbb{N}$ *is r.e. but not recursive, then*

$$G = \langle a, b, t \mid [t, b^n a b^{-n}] = 1 \ \forall n \in S \rangle$$

has an unsolvable word problem*.*

Proof: The $a_n := b^n a b^{-n}$ generate a free group, and HNN normal form says $w_m := [t, b^m a b^{-m}]$ equals $1 \in G$ if and only if $m \in S$. □

Higman Embedding Theorem: *A f.g. group* G *can be recursively presented iff there is a finitely presented group* $\Gamma \supseteq G$*.*

Corollary 7.3 (Novikov, Boone)**.** *There exist finitely presented groups with unsolvable word problems.*

Proof: Embed G from Proposition 7.2 in a finitely presented group Γ. If one could solve the word problem in Γ then one could decide which words in the generators of G represented the identity.

Having shown the word problem to be undecidable for finitely presented groups in general, it is easy to encode this undecidability into all manner of other problems...

7.3. **Isomorphism Problem:** Let $\Gamma = \langle A \mid R \rangle$ be a finitely presented group with an unsolvable word problem. Write $A = \{a_1, \ldots, a_n\}$. Assume each $a_i \neq 1$ in Γ. For each word $w \in A^*$, define

$$\Gamma_w = \langle A, s, t \mid R, \quad t^{-1}(s^{-i} a_i s^i) t = s^{-i} w s^i \ \forall i \rangle.$$

Then Γ_w is a free group of rank 2 if and only if $w = 1$ in Γ. Thus

Corollary 7.4. *The* isomorphism problem *for finitely presented groups is unsolvable.*

In a similar manner one quickly deduces:

Corollary 7.5. *The* triviality problem *for finitely presented groups is unsolvable.*

7.4. **Isomorphism of Manifolds.** Recall that every finitely presented group can be realised as the fundamental group of a closed 4-manifold. By following a standard proof of this construction *with considerable care* one can prove:

Theorem 7.6. [**Markov**] *In dim* ≥ 4*, there is no algorithm to decide* isomorphism *among arbitrary closed (smooth, topological or PL)* manifolds.

Are we to give up on deciding isomorphism among manifolds then? NO! Topologists never give up!

We know that if we are to hope for positive results then we must make a restriction of some sort on the class of manifolds under consideration. The most obvious restriction is dimension.

Conjecture 7.7. *The homeomorphism problem is* solvable among *compact* 3–manifolds.

Conjecture 7.8. *The isomorphism problem is solvable among 3–manifold groups.*

Conjecture 7.9. *The word and conjugacy problems are solvable for all 3–manifold groups.*

All of these statements are non-trivial consequences of Thurston's Geometrization Conjecture (which at the time of writing looks close to becoming a theorem of Perelman).

7.5. **Isomorphism of Negatively Curved Groups & Manifolds.** Instead of restricting to low-dimensional manifolds, one might attempt to get positive results concerning isomorphism of manifolds by placing *geometric constraints* of the manifolds considered. And in the light of our earlier discussion showing the natural emergence of negative and non-positive curvature in the setting of the word problem for groups, it seems natural to look in this direction again, particularly in the light of Farrell and Jones's remarkable work on *Topological Rigidity.* Here is a weak form of what they prove:

Theorem 7.10. [**Farrell-Jones**] *Let* $n \geq 5$ *and let* M *and* N *be closed non-positively curved manifolds. If* $\pi_1 M \cong \pi_1 N$*, then* $M \cong N$ *(homeomorphic).*

A Mayer–Vietoris argument shows that π_1 of a closed aspherical M^n $(n \geq 3)$ cannot split as $A * B$, or $A *_{\mathbb{Z}} B$, or $G*_{\mathbb{Z}}$. Moreover, $\pi_1 M$ is torsion-free. Thus we may combine the above result with:

Theorem 7.11. [**Sela**] *The isomorphism problem is solvable in the class of torsion–free hyperbolic groups that do not split over* $\{1\}$ *or* $\mathbb{Z}$*.*

These two wonderful theorems together imply:

Theorem 7.12. *If* $n \geq 5$*, then the homeomorphism problem is solvable in the class of closed manifolds that admit a metric of* strictly negative curvature.

8. Isomorphism & Undecidability around NPC Groups and Manifolds

Note that the Farell–Jones result applies to non-positively curved manifolds, whereas Sela's result does not. Having thought about this for some time, I now believe:

Conjecture 8.1. *The homeomorphism problem is unsolvable amongst non–positively curved manifolds.*

This conjecture is open, but I shall present some partial results in this direction.

8.1. **Isomorphism Problems around NPC Groups.**

I shall sketch the proofs of the following theorems.

Theorem 8.2 (B, 2001). *The* isomorphism *problem is* unsolvable *in the class of combable groups.*

Theorem 8.3 (B, 2001). *The* conjugacy *problem is* unsolvable *in certain combable groups.*

8.2. **A seed of undecidability.**

Proposition 8.4. *There is an integer k and a recursive sequence of finite subsets $S_n \subset F_2 \times F_2$, with $|S_n| = k$, such that there is no algorithm that will answer the following questions correctly for all n:*

(1) Does S_n generate $F_2 \times F_2$?
(2) Is $\langle S_n \rangle$ finitely presented?

Idea [following Mihailova and Miller in the 1970s]: Associated to $Q = \langle a_1, \dots, a_l \mid r_1, \dots, r_m \rangle$ one has a s.e.s.

$$1 \to N \to F_l \xrightarrow{\pi} Q \to 1$$

and *fibre product*

$$P := \{(u,v) \mid \pi(u) = \pi(v)\} \subset F_l \times F_l.$$

Lemma 8.5. *P is generated by*

$$\{(r_1, 1), \dots, (r_m, 1), (a_1, a_1), \dots, (a_l, a_l)\}.$$

Lemma 8.6. *P is finitely presented iff Q is finite.*

Proof of Prop: Take a sequence of presentations Q_n, each with l generators and m relations, such that there is no algorithm to determine which Q_n are finite or trivial.

Define $S_n \subset F_l \times F_l$ to be the generators for the corresponding fibre product.

$\langle S_n \rangle = F_l \times F_l$ iff $Q_n = \{1\}$

$\langle S_n \rangle$ is f.p iff $Q_n = \{1\}$

Theorem 8.7. *The* isomorphism *problem is* unsolvable *among the sequence of* combable *groups $\hat{\Delta}_n$ described below.*

Crude Idea: Let Γ be a CAT(0) group (such as $F \times F$) in which one has a sequence of finite subsets (S_n), as above. Arrange $\langle S_0 \rangle = \Gamma$. Consider the natural maps

$$\phi_n : F_k \twoheadrightarrow \langle S_n \rangle \subset \Gamma$$

In $G = \Gamma \times F_k$, consider the *retract*

$$\Sigma_n = \{(\phi_n(x), x) \mid x \in F_k\}$$

and define

$$\Delta_n = G *_{\Sigma_n} G.$$

The crude idea is that since one can't decide much about the subsets S_n and hence the maps ϕ_n, by doubling along the graphs Σ_n of the ϕ_n we ought to have obtained a sequence of groups with an element of undecidability. On the other hand, since Σ_n is a retract it is isometrically embedded (with a suitable choice of word metrics), so by analogy with the process of doubling a non-positively curved space along a totally-geodesic subspace, one expects the groups $G *_{\Sigma_n} G$ to be NPC (to some extent at least).

Like most crude ideas, this does not work. But like the best crude ideas, it contains a kernel of wisdom that can be adapted to give the desired construction.

Refinement: Mapping the extra basis elements trivially, replace ϕ_n by

$$\hat{\phi}_n : F_{2k} \to \langle S_n \rangle \subset \Gamma,$$

and consider

$$\hat{G} = \big(A * \Gamma\big) \times F_{2k},$$

where $A \not\to \Gamma$ has Serre's property FA, and

$$\hat{\Delta}_n = \hat{G} *_{\hat{\Sigma}_n} \hat{G}.$$

The conjugacy problem:

Let me now sketch how one constucts combable groups with unsolvable conjugacy problem. We follow the construction of Δ_n as above, but instead of taking a sequence of maps $F_k \to \Gamma$, we take a single map ϕ such that membership of $\mathrm{im}(\phi)$ is undecidable, using (8.4). We also arrange that there exist $a \in \Gamma$ whose centralizer $Z_\Gamma(a)$ lies in the image of ϕ.

Thus we have

$$\Delta = (\Gamma \times F_k) *_\Sigma \overline{(\Gamma \times F_k)},$$

and we have arranged things so that there is no algorithm to decide the following in Δ

$$\text{given } b \in \Gamma \text{ is } (bab^{-1})\overline{(bab^{-1})} \text{ conj to } a\overline{a}?$$

On the other hand, one can prove that if Γ is semihyperbolic and F is combable then Δ is combable.

8.3. **Baumslag-B-Miller-Short, CMH 2000.** Finitely presented subgroups of hyperbolic groups appear to be well behaved. Indeed, there is a only a single example of such a subgroup that is not itself hyperbolic (due to Noel Brady) and I proved that this has a solvable conjugacy problem. On the other hand, we have seen that finitely generated subgroups of hyperbolic groups can be rather badly behaved, and the following results show that the same is true of the finitely presented subgroups of hyperbolic groups.

Theorem 8.8. *$\exists$ hyperbolic Γ and f.p. $P \subset \Gamma \times \Gamma$ s.t.* membership of P is undecidable *and* conjugacy *problem for P is* unsolvable. *(One can arrange for Γ to be the fundamental group of a compact negatively curved 2-complex.)*

Theorem 8.9. *$\exists$ closed* non-positively curved manifold M^9 *and a recursive class of finitely presented subgroups of $\pi_1 M$ s.t. there is* no algorithm *to determine* homotopy equivalence *between the covering spaces corresponding to these subgroups.*

The original proof of this last theorem can be simplified considerably using a later result of Miller and myself (PAMS 2003):

Theorem 8.10. *If $1 \to K \to \Gamma \to Q \to 1$ be a s.e.s of groups with Γ torsion-free hyperbolic, Q free non-abelian, and K infinite and finitely generated, then there is no algorithm to decide isomorphism among the finitely presented subgroups of $\Gamma \times \Gamma \times \Gamma$.*

8.4. **Subgroups of Direct Products.**

Theorem 8.11 (Stallings-Bieri)**.** *The kernel SB_n of $F_2 \times F_2 \to \mathbb{Z}$ is of type $\mathcal{F}_{n-1}$ but not of type $\mathcal{F}_n$.*

Reason: Double mapping-cylinder and MV on

$$\mathrm{SB}_n \cong F_2 *_{\mathrm{SB}_{n-1}} F_2$$

These examples have given rise to several beautiful threads of research in geometric group theory. Bieri's investigations led him further (with Strebel, later Neumann, and many others) into a detailed study of higher finiteness properties of groups, as described in the first lecture. It was also through their attempts to understand the geometry behind the above example that Bestvina and Brady discovered their elegant combinatorial Morse theory, which allowed them to solve an old problem by constructing groups of type $\mathrm{FP}_2(\mathbb{Z})$ that are not finitely presented.

In this lecture I want to pursue a third strand of mathematics that begins with the above examples, namely the pursuit of a complete understanding of the (apparently wild) finitely presented subgroups of free and related groups. A note of dissonance with the expectation of wildness was struck with Baumslag and Roseblade when they proved the following theorem:

Theorem 8.12 (Baumslag-Roseblade)**.** *If $S \subset F^{(1)} \times F^{(2)}$ is fin pres, either S is* free *or $S \supset L_1 \times L_2$, finite index, with $L_i \subset F^{(i)}$.*

The next step in the programme came about ten years later. The following theorem is due to Bridson-Howie-Miller-Short. Here, *"surface group" means π_1(compact surface)* (allowing non-empty boundary).

Theorem 8.13. *Σ^i surface groups, then $S \subset \Sigma^1 \times \cdots \times \Sigma^n$ is type $\mathcal{F}_n$ if and only if S is itself virtually a product of (at most n) surface groups.*

Finer information: Replacing the Σ_i by the projections of S, one may as well assume that S is a sub-direct product; and projecting away from direct factors where S has trivial intersection, one may also assume that each $L_i := S \cap \Sigma_i$ non-trivial. Suppose that m of the L_i are finitely generated and $n - m$ are not. What we then prove is that S contains a subgroup of finite index $S_0 = S_1 \times \cdots \times S_m \times B$, where the S_i are surface groups and $H_{n-m}(B, \mathbb{Z})$ is not finitely generated.

Idea of Proof: One uses various facts about subgroups of surface groups, for example a version of Marshall Hall's theorem: every finitely generated subgroup of a surface group is a retract. This relies on a weak from of Scott's *surf-lerf* theorem: cyclic subgroups are closed in the profinite topology.

The main argument is an induction on n, using the LHS spectral sequence to analyse a projection onto a direct factor. In the homology calculation, one needs the Fox free calculus.

Remark 8.14. Jim Howie and I have recently extended the above theorem into the realm of *limit groups*. This involves showing that limit groups share many properties of surface groups, particularly with regard to subgroup structure. For example, if the first homology of a subgroup is finitely generated then so is the subgroup, and if the limit group is non-abelian, then its only non-trivial finitely generated normal subgroups are those of finite index.

On the other hand, the theorem does not extend to products of arbitrary 2-dimensional hyperbolic groups (even small-cancellation groups), and in a moment we shall see that it does it extend to products of higher-dimensional Kleinian groups.

Of course, the above theorem leaves open the question of just how diverse the finitely presented (not FP_n) subgroups of a direct product of n free groups may be. Work of Meier, Meinert and van Wyck gives precise information about normal subgroups with abelian quotients (this is one of triumphs of the Bieri approach), and the following result of Miller and myself suggests that this seemingly specialised work is much more generic than one would expect:

Theorem 8.15 (B-Miller)**.** *If f.p. $S \subset F \times F \times F$ intersects each factor and projects onto each factor, S is either virtually a direct product of free groups, or else* contains a subgroup of finite index that it normal in $F \times F \times F$ with abelian quotient.

We have similar results for any number of factors, but it is unclear at the moment what the best possible theorem in this direction is. Our results suggest that the isomorphism and conjugacy problems are solvable among finitely presented subgroups of $F \times \cdots \times F$, but we have not yet written this out in detail.

8.5. **Bridson's Remarkable** $(3n-1)$**-manifolds!** Please forgive the sarcastic name: these manifolds are great and deserve to be better known. Take a hyperbolic 3-manifold $\Sigma \to M^3 \to \mathbb{S}^1$. Take Cartesian sum of n copies and consider

$$N^{3n-1} \to M^{3n} \to \mathbb{S}^1$$

Then N is a closed aspherical $(3n-1)$-dimensional manifold and $\pi_1 N$ displays *almost all* the properties of $\pi_1(\text{NPC})$, but N does not support a metric of non-positive curvature.

Indeed no subgroup of finite index in $\pi_1 N$ is semihyperbolic. This shows in particular that the [BHMS] theorem does not extend to higher-dimensional manifolds.

These examples are explained in detail in *Subgroups of semihyperbolic groups*, Haefliger Festschrift, Monog. L'Enseign. Math., 2001.

8.6. **Subgroups in low dimensions.** We have had various results to the effect that subgroups of non-positively curved groups can be fiendishly complicated. Let me offset this a little by pointing out that there are positive results in low dimensions; these results are proved in the above article.

For 2-dimensional complexes a tower argument allows one to prove:

Theorem 8.16. *If* Γ *is the fundamental group of a 2-dimensional non-positively curved complex, then every finitely presented subgroup of* Γ *is the fundamental group of a compact, 2-dimensional, non-positively curved complex.*

A more complicated argument involving Thurston's geometrization theorem for Haken manifolds allows one to prove:

Theorem 8.17. *If* Γ *is the fundamental group of a compact 3-dimensional non-positively curved manifold (possibly with boundary), then every finitely presented subgroup of* Γ *is also the fundamental group of such a manifold.*

Lecture 3: Encoding Problems from Elsewhere into NPC

The main point of this lecture is to exemplify the approach articulated at the end of lecture 1: we want to encode problems from elsewhere in mathematics into questions about (perhaps wild) finitely presented groups; we then try to tame the situation by translating the salient properties of the groups in question into properties of non-positively curved groups, where we hope to solve the problem at hand.

In this lecture I want to explain two techniques for constructing interesting finitely presented groups — the Rips construction and the 1-2-3 Theorem. I shall sketch how, by using these constructions in tandem with others, Fritz Grunewald and I were able to answer the question of Grothendieck stated at the end of lecture 1.

Let me begin with a short story that brought Fritz and I together.

9. A Short Story about GL$(n, \mathbb{Z})$

The following is a classical theorem of Borel & Harish-Chandra:

Theorem 9.1. *Arithmetic groups contain only finitely many conjugacy classes of finite subgroups.*

It is a strikingly non-trivial matter to improve this result to finite extensions of arithmetic groups, but such an extension was established by Grunewald and Platonov in 2000. In the course of their work, they also solved a problem posed by Platonov in 1968:

Theorem 9.2. *There exist finitely generated groups $H \subset GL(n, \mathbb{Z})$ with infinitely many conjugacy classes of finite subgroups.*

They then posed the following problem: *"The question remains whether $GL(n, \mathbb{Z})$ contains a finitely presented subgroup with infinitely many conjugacy classes of finite subgroups."*

I solved this problem in a paper that appeared in Math. Ann. in 2001:

Theorem 9.3. *There exist $H_n \subset GL(2n+2, \mathbb{Z})$ of type $\mathcal{F}_n$ with* infinitely many conjugacy classes of finite subgroups.

My construction is based on the well-known epimorphism to $\mathbb{Z}$ from $\Gamma_0(17) \subset \mathrm{SL}(2, \mathbb{Z})$, the group of matrices where 17 divides the bottom-left entry. By taking the direct sum of n copies of this group and adding the epimorphisms to $\mathbb{Z}$ one gets the desired examples

$$H_n = \ker(\Gamma_0(17) \times \cdots \times \Gamma_0(17) \to \mathbb{Z}).$$

One can construct subgroups of direct products of $\mathrm{PSL}(2, \mathbb{R})$ with infinitely many conjugacy classes of finite subgroups by replacing $\Gamma_0(17)$ with

$$\langle a, b \mid [a, b]^p = 1 \rangle.$$

Ian Leary and Brita Nuncinkis later constructed subgroups of $\mathrm{GL}(n, \mathbb{Z})$ which are of type $\mathcal{F}_\infty$ and contain infintiely many conjugacy classes of finite subgroups. Their construction is based on a different circle of ideas beginning with the Morse theory of Bestvina and Brady that I referred to earlier.

10. Fibre Products and the 1-2-3 Theorem

Associated to any short exact sequence $1 \to N \to H \xrightarrow{\pi} Q \to 1$ one has a *fibre product*

$$P := \{(h_1, h_2) \mid \pi(h_1) = \pi(h_2)\} \subset H \times H.$$

Let $N_1 = N \times \{1\}$, let $N_2 = \{1\} \times N$ and $\Delta = \{(h,h) \mid h \in H\} \cong H$. One has the following semi-direct product decompositions:

$$P = N_1 \cdot \Delta \; = \; N_2 \cdot \Delta \; \cong \; N \rtimes H,$$

where the action is simply conjugation.

Lemma 10.1. *If H is finitely generated and Q finitely presented, then P is finitely generated.*

This is an easy exercise.

The question of when P is finitely presented is much more subtle. To get some idea of why, note that if N not finitely generated then one might anticpate needing infinitely many relations to force N_1 to commute with N_2.

Baumslag-Bridson-Miller-Short studied the issue of finite-presentability in detail in order to prove the theorems in [CMH 2000] that I talked about yesterday.

Theorem 10.2 ("1-2-3 Theorem"). Suppose that $1 \to N \to \Gamma \xrightarrow{p} Q \to 1$ is exact and consider the fibre product

$$P := \{(\gamma_1, \gamma_2) \mid p(\gamma_1) = p(\gamma_2)\} \subset \Gamma \times \Gamma.$$

If N is finitely generated, Γ is finitely presented and Q is of type $\mathcal{F}_3$, then P *is finitely presented.*

The name of this theorem derives from the hypothesis: N, Γ and Q are assumed to be of type $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ respectively.

To see how the condition $\mathcal{F}_3$ enters, note first that *relations among the relations in Q give correspond to 2-spheres in $K(Q,1)$*. (This issue has re-emerged at crucial points throughout the history of low-dimensional homotopy theory, in particular J.H.C. Whitehead's theory of π_2 and crossed modules, and Alan Turing's brief incursion into combintorial group theory in the 1950s.)

2-spheres in $K(Q,1)$ lead to relations among the generators of N in the following way: if $\underline{a}$ is a set of choices of lifts $a_i \in \Gamma$ of the generators of Q, and $\underline{x}$ is a set of generators of N, then since N is normal,

$$a_i^{-1} x_j a_i =_P U_{ij}(\underline{x})$$

for some word U_{ij}, and since $p(r(\underline{a})) =_Q 1$ for each relation r of Q,

$$r(\underline{a}) =_P V_r(\underline{x}).$$

Conjugating x_j with $r(\underline{a})$ using the first set of relations leads to a relation of the form

$$V_r^{-1} x_j V_r = R_{jr}(\underline{x}).$$

If $r_1 = r_2 r_3$ (freely), we get further relations among the generators $\underline{x}$. Geometrically, an equality such as $r_1 = r_2 r_3$ corresponds to an element of $\pi_2 K(Q, 1)$ represented by the spherical diagram formed by joining three discs with boundary labels r_1, r_2, r_3.

11. The Rips Construction

In order to make the 1-2-3 Theorem a powerful tool, one needs a rich supply of input s.e.s. $1 \to N \to \Gamma \to Q \to 1$. Such a supply is provided by the following construction of E. Rips.

Theorem 11.1. *There exists an algorithm with input a finite presentation $\mathcal{Q}$ and output a s.e.s.*

$$1 \to N \to \Gamma \to Q \to 1$$

where N is a 2-generator group, Γ is a 2-dimensional hyperbolic group, and $Q = |\mathcal{Q}|$.

Key point: One is fattening-up the (1-dimensional) free group implicit in the presentation of Q to a 2-dimensional hyperbolic group; the pay-off is *finite generation* of N

Idea of proof: If $Q = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$, take $N = \langle a, b \rangle$, and as generators of Γ take $x_1, \dots, x_n, a, b$.

The relations of Γ are:

$$r_i(\underline{x}) = U_i(a, b) \quad i = 1, \dots, m$$
$$x_j^{\epsilon} a x_j^{-\epsilon} = V_{j\epsilon}(a, b) \;\; x_j^{\epsilon} b x_j^{-\epsilon} = W_{j\epsilon}(a, b)$$

where the words $U_i, V_{j\epsilon}, W_{j\epsilon}$ are long words satisfying a *small cancellation condition.*

UTILITY: The Rips construction encodes arbitrary finite presentations into s.e.s. with hyperbolic Γ. (One can then use the *1-2-3 Theorem* to turn the s.e.s. into a *finitely presented* subgroup $P \subset \Gamma \times \Gamma$.)

We saw in lecture 2 how to encode arbitrary degrees of pathology into finite presentations of groups; the Rips construction pushes this pathology into the pair (N, Γ), whence the associated fibre product.

PRACTICAL: If the input presentation $\mathcal{Q}$ is aspherical, the passage from $\mathcal{Q}$ to P is *entirely algorithmic.*

There is a huge amount of flexibility in this construction: If you choose long words randomly, they will work. Or you can make careful choices to ensure that the group has extra properties.

By exploiting the flexibility in the choice of small cancancellation words, one can enhance the Rips construction to ensure that the group Γ has various additional properties. For example:

Theorem 11.2. *There is an algorithm that associates to any finite group presentation $\mathcal{Q}$ a compact, negatively curved, piecewise hyperbolic 2-dimensional complex K and a short exact sequence*

$$1 \to N \to \pi_1 K \to Q \to 1,$$

$Q = |\mathcal{Q}|$, *N f.g. by $\underline{a}$, each $a \in \underline{a}$ is represented a 1-cell in K that is a closed geodesic.*

The most important such improvement from the point of view of what we are doing here is due to Dani Wise:

Theorem 11.3 (Rips-Wise)**.** *There is an algorithm that associates to every finite group presentation $\mathcal{P}$ a short exact sequence of groups*

$$1 \to N \to H \to Q \to 1,$$

where Q is the group presented by $\mathcal{P}$, the group N is generated by three elements, and the group H is a torsion-free, residually-finite, hyperbolic group (satisfying the small cancellation condition $C'(\frac{1}{6})$).

12. Grothendieck's Question

The profinite completion of a group Γ is the inverse limit of the directed system of finite quotients of Γ; it is denoted $\hat{\Gamma}$.

Question 12.1. *Let Γ_1 and Γ_2 be finitely presented, residually finite groups, and $u : \Gamma_1 \to \Gamma_2$ a homomorphism such that $\hat{u} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$ is an isomorphism of profinite groups. Does it follow that u is an isomorphism?*

We proved the following in the spring[7] of 2003:

Theorem 12.2. *There exist residually finite, 2-dimensional, hyperbolic groups H and finitely presented subgroups $P \hookrightarrow \Gamma := H \times H$ of infinite index, such that P is not abstractly isomorphic to Γ, but the inclusion $u : P \hookrightarrow \Gamma$ induces an isomorphism $\hat{u} : \hat{P} \to \hat{\Gamma}$.*

In the remainder of the lecture, I'll sketch the proof. You will see that the ideas from each of the preceding sections plays a crucial role.

12.1. A Platonov-Tavgen Type Criterion.

Theorem 12.3. $1 \to N \to H \to Q \to 1$ *exact, with fibre product $P \subset H \times H$. If H f.g., Q has no finite quotients, and $H_2(Q, \mathbb{Z}) = 0$, then $P \hookrightarrow H \times H$ induces an isomorphism. $\hat{P} \to \hat{H} \times \hat{H}$.*

[7]appeared 2004

Surjectivity: We must show that there is no proper subgroup of finite index $G \subset \Gamma$ containing P. If there were, $N \times N \subset P \subset G$, and $G/(N \times N)$ would be a proper subgroup of finite index in $(H/N) \times (H/N)$, *contradiction.*

Injectivity: This is more delicate. One must show that for each normal $R \subset P$ of finite index, there exists $S \subset \Gamma$ of finite index with $S \cap P = R$.

The hypothesis that Q has no finite quotients enters several times, once with the observation that any extension

$$1 \to F \to G \to Q \to 1$$

with F finite gives rise to a *central* extension

$$1 \to F \cap G' \to G' \to Q \to 1$$

with $G' \subset G$ of finite index.

Since Q is perfect, it has a *universal* central extension

$$1 \to H_2(Q, \mathbb{Z}) \to \tilde{Q} \to Q \to 1.$$

And the hypothesis $H_2(Q, \mathbb{Z}) = 0$ ensures that every central extension of Q splits.

12.2. **Balanced Presentations.** The strategy for proving Theorem 12.2 is as follows: find a superperfect group Q of type $\mathcal{F}_3$ that has no finite quotients, feed it through the Rips-Wise construction and the 1-2-3 Theorem to obtain a finitely presented residually finite pair $P \hookrightarrow \Gamma \times \Gamma$, then apply the criterion established in the preceding sub-section.

In order to implement this strategy we need input groups Q. We obtain them by looking at suitable balanced presentations, i.e. presentations

$$\langle a_1, \ldots, a_n \mid r_1, \ldots, r_n \rangle$$

with equal numbers of generators and relations.

This fascinating condition has a high degree of topological import (cf. Andrews-Curtis conjecture, fundamental groups of 4-dimensional homology spheres, etc.).

Proposition 12.4. *There exist infinite groups Q, given by finite, aspherical, balanced presentations, such that Q has no non-trivial finite quotients.*

Lemma 12.5. *If Q has a balanced presentation and $H_1(Q, \mathbb{Z}) = 0$, then $H_2(Q, \mathbb{Z}) = 0$.*

One can build examples from $\{1\}$ by amalgamated free products and HNN extensions along free subgroups.

The Higman Groups

Higman constructed the first examples: $n \geq 4$,

$$J_n = \langle a_1 \ldots, a_n \mid a_2^{-1} a_1 a_2 a_1^{-2}, \ldots, a^{-1} a_n a_1 a_n^{-2} \rangle.$$

Here are some new examples:

Amalgamating Non-Hopfian Groups

Fix $p \geq 2$ and consider

$$G = \langle a_1, a_2 \mid a_1^{-1} a_2^p a_1 = a_2^{p+1} \rangle.$$

G has a non-injective epimorphism

$$\phi(a_1) = a_1,\ \phi(a_2) = a_2^p.$$

Observe:

- $c = [a_2, a_1^{-1} a_2 a_1]$ is in kernel of ϕ
- $\forall \pi : G \to$ finite one has $\pi(c) = 1$ (o.w. $\pi \circ \phi^n$ would all be different, which is impossible).
- $\langle a_1, c \rangle \cong F_2$ in G
- G modulo $\langle\langle a_1 \rangle\rangle$ is $\{1\}$.

Proposition 12.6. *If* $p \geq 2$*,* the aspherical

$$\langle a_1, a_2, b_1, b_2 \mid a_1^{-1} a_2^p a_1 a_2^{-p-1},\ b_1^{-1} b_2^p b_1 b_2^{-p-1},\ a_1^{-1}[b_2, b_1^{-1} b_2 b_1],\ b_1^{-1}[a_2, a_1^{-1} a_2 a_1] \rangle$$

has no non-trivial finite quotients.

12.3. **Counterexamples to Grothendieck's Question.** Take an aspherical, balanced presentation of a group with no finite quotients, Q, apply the Rips-Wise construction to get a short exact sequence

$$1 \to N \to H \to Q \to 1$$

with H residually finite and hyperbolic group and N finitely generated.

The 1-2-3 Theorem says the fibre product $P \subset H \times H$ is finitely presented, and since Q is infinite P is a subgroup of infinite index.

The sequence $1 \to N \to H \to Q \to 1$ satisfies the Platonov-Tavgen type criterion, hence

$$P \hookrightarrow H \times H$$

induces an isomorphism $\hat{u} : \hat{P} \to \hat{H} \times \hat{H}$. □

13. Epilogue

We began by asking why group presentations are a natural way to consider groups being *given*. The first steps in decoding the information present in a finite presentation of a group lead one immediately to the *word problem*, and further thought brings one to the other fundamental decision problems (à la Dehn). If one maps the universe of group theory according to the complexity of the the word problem, hyperbolic groups emerge in a canonical and key role, followed by the cloudier notion of non-positively curved groups (NPC), the very definition(s) of which involve subtle distinctions, not least in the tractability of the basic decision problems in this area of the group-theoretic universe.

This is where we stand as we attempt to understand groups: we stand in the region of NPC, with the safe but exciting world of negative curvature behind us, and the wildness of arbitrary finitely presented groups before us: *between the sea and the sky, where the weather is.*

We saw how, via Higman's Embedding Theorem, one can readily encode undecidable problems into finite group presentations and compact manifolds in dimension 4 and above. On the other hand, the basic decision problems become tractible in the presence of negative curvature (and for 3-dimensional manifolds). And we find that NPC provides a fascinating frontier environment, where the limits of decidability are yet to be determined.

We pursued the idea that one can profitably stand in the region NPC and reach out into the wild yonder of arbitrary finitely presented groups, dragging the arbitrary pathologies of that theory into NPC by various constructions, in particular encodings into the subgroup structure of apparently benign groups. As a result of such techniques one can bring the tools of non-positive curvature to bear on an array of group-theoretic and topological problems. (In particular one has local-to-global results, both in the sense of the Cartan-Hadamard Theorem [with extensions into complexes of groups, etc.] and in more combinatorial senses, e.g. the paradigm of Cannon's insights into the cone types of hyperbolic groups.)

We exemplified this *drawing-in* philosophy by studying the recent solution by Bridson and Grunewald of a problem of Grothendieck concerning representation theory and isomorphism among profinite groups.