Geometric constructions and elements of Galois' theory
List 1

Constructible numbers and algebraic methods in geometric constructions.

1. Given a unit segment (i.e. a segment of length 1), construct segments of lengthgs $\frac{1}{5}$, $\frac{11}{17}$, $\sqrt{2}$, $\sqrt{2\frac{1}{3}}$, $\sqrt{1+\sqrt{2}}$, $\sqrt{\sqrt{5}-\sqrt{3}}$, $\frac{2+\sqrt{7}}{\sqrt{3}-\sqrt{2}}$, $\sqrt[8]{2}$.
   Describe steps of these constructions using schematic pictures.
2. Given points $(0,0)$ and $(1,0)$ on the plane, construct points $(2\frac{2}{3}, -1\frac{1}{7})$, $(-5\frac{1}{2}, \frac{1}{2}+\frac{1}{3}\sqrt{5})$.
3. Given a segment of length 1, construct a square for which the sum of its perimeter and its area equals 4.
4. Construct an isosceles (równoramienny) triangle, having given its perimeter $2p$ and its height $h = 1$.
5. We are given the segments a, b, c, d, e, f. Supposo also that the unit vsegment is not given. Construct segments with the following lengths: $\frac{ab}{c}$, $\frac{2a^2}{3b}$, $\frac{abc}{df}$, $\frac{a^2+2bc}{d}$, $\frac{ab}{3c+d}$, $\frac{a^2+2bc}{d+f}$, $\sqrt{\frac{1}{2}ab}$, $\sqrt{a^2+b^2}$, $\sqrt{c^2-d^2}$, $\sqrt{a^2+bc}$, $\sqrt{a^2+3b^2}$, $\sqrt{ab-\sqrt{2}cd}$, $a\sqrt{2}$, $a\sqrt{7}$, $a\sqrt{3}/(1+\sqrt{5})$, $\sqrt[4]{abcd}$, $\sqrt[4]{a^4-b^4}$, $\sqrt[4]{2a^4+3b^4}$.
6. We are given the segments a and b. Construct two segments whic satisfy the fo9llowing
   (a) their sum is a and the product of their lengths is b^2;
   (b) their difference is a, and the product of their lengths is b^2.
7. Suppose we are given points A and B, whose coordinates are constructible numbers, and let O denotes the origin of the coordnate system. Prove that the following numbers are trhen constructible:
   a. the length of the segment AB,
   b. cosine of the angle AOB,
   c. area of the triangle AOB.
8. Suppose we are given an angle α, whose cosine is a constructible number. Show that the following numbers are then also constructible:
   a. $\sin\alpha$; b. $\cos\frac{1}{2}\alpha$;
   c. length of a chord of a unit circle which is a base for an inscribed angle of size α.
9. For an arbitrary rectangle (which is not a square) divide its longer side, constructibly, into such two pieces, so that the sum of areas of squares built upon those pieces is equal to the area of the rectangle.
10. Describe construction of the **golden cut** of a given segment. Recall that a point C proviedes the golden cut of the segment AB if $AC : CB = AB : AC$.
    Hint: view the given segment as a unit (i.e. as if it has length 1), calculate algebraically the lengths of AC and BC, and give a schematic sketch of the required construction.
11. (a) Verify, by calculating angles and detecting some auxiliary isosceles triangles, that for any two intersecting diagonals in a regular pentagon the intersection point cuts out of each of them a segment of the same length as side of the pentagon.
    (b) Show that the intersection point of any two diagonals as above cuts each of the diagonals in the golden proportion.
    (c) Describe (schematically) a construction of a regular pentagon based on the above observations, and on the construction of the golden cut.

12. a. Let c' be the length of the chord (in a unit circle) corresponding to the central angle whose measure is half of the measure of the central angle corredsponding to the chord of length c. Show that $c' = \sqrt{2 - \sqrt{4 - c^2}}$.
    b. Derive inductively a formula for the side length of the regular 2^n-gon and the regular $3 \cdot 2^n$-gon inscribed in a unit circle. Show that this length is a constructible number. Could one predict this last phenomenon without calculations?
13. Using an algebraic approach, show that the following constructions are possible using compass and straightedge:
    (a) construction of a square whose area is equal to the area of a given (arbitrary) triangle (squaring of a triangle);
    (b) construction of the circle, whose are equals the sum of areas of any two given circles (we view a circle as given, if we are given its center and its radius);
    (c) splitting of any given triangle into two parts of equal area by a line parallel to any of the sides of this triangle.

    Hint: fix notation for parameters of given objects, viewing some of these parameters as a unit segment; then reduce the required construction to the construction of some single segment; finally, calculate the length of this demanded segment and show algebraically that it can be constructed.
14. View the points $(0, 0)$ oraz $(1, 0)$ as given. Verify algebraically, by performing appropriate calculations, that the following is possible: construction of the circle tangent to the circle $x^2 + y^2 = 1$ and passing through the points $A(2, 1)$ and $B(3, 2)$. Hint: check that the coordinates of the center and the radius of the required circle are all some constructible numbers.