Geometric constructions and elements of Galois' theory
List 3

Roots of polynomials of degree 3. Impossible constructions.

1. Verify wheather the following polynomials have rational roots:
   a. $x^2 - 4x - 1$;
   b. $x^3 - x^2 - 13x - 3$;
   c. $4x^3 - 4x + 1$;
   d. $8x^4 - 4x^3 - 8x^2 - 2x - 1$.
   Do they have constructible roots?
2. Suppose we are given the unit interval. Is it possible to cunstruct a cube, for which the sum of its area and its volume equals: (a) 5; (b) 3? If "yes", then is it possible to construct **all** cubes with this property?
3. Given a segment of length 4, is it possible to divide it, using a geometric construction, into three segments, two of which are of the same length, and so that the p-erpendicular built upon those three segments has volume 1?
4. Is it possible to construct geometrically an isosceles triangle whose legs have length 5, and for which the radius of the inscribed circle equals 1? Solve this by performing the following steps of reasoning:
   a. Denote by d half of the base length in this triangle and show that, in order to construct such a triangle, it is necessary and sufficient to construct the segment of length d.
   b. Find a polynomial with rational coefficients which has d as one of its roots. To do this, identify (through an equality) two expressions for the square of area of this triangle, one obtained by the formula $P = pr$, and the other by the formula $P = \sqrt{p(p-a)(p-b)(p-c)}$, where p denotes the half of the perimeter of the triangle, a, b, c are the lengths of its sides, and r denotes the radius of tits inscribed circle. You will obtain a polynomial F of degree 4.
   c. Find a rational root q of the polynomial F obtained in the previous step, verify that this root is not equal to d, divide F by the polynomial $x - q$, and analyze the so obtained new polynomial of degree 3.
5. Is it possible to construct geometrically an isosceles triangle whose angle bisectors have lengths 1, 1 and 2? To solve this exercise perform the following steps:
   a. Denote by α half of the angle near the base of the required triangle and show that, in order to construct this triangle it is necessary and sufficient to construct a segment of length $\sin \alpha$.
   b. Express the length of the base AB of the required triangle in terms of α, using the function tangent (pol. tangens) applied to the triangle obtained by bisecting the required triangle along its mirror symmetry axis.
   c. Express the langth of the base AB differently, using the law of sines (pol. twierdzenie sinusów) applied to the sides AB and AD of the triangle ABD, where D is the second end of the angle bisector of the required triangle started at the vertex A.

d. Form equality out of the two above ontained expressions for $|AB|$ (in steps b. and c.). Express all appearing ingredients which involve trigonometric functions in terms of $\sin\alpha$. Transform the obtained eqaulity to the form in which $\sin\alpha$ turns out to be the root of some polynomial of degree 3.

e. Examine the so obtained polynomial, as far as constructibility of its roots.

6. Is it possible to perform gemetrically the operation of splitting of arbitrary angle into 4, 5, 6, 7, 8, 9 equal angle parts?
7. Is it possible to construct geometrically the angles of measures 10°, 5°, 15°, 25°, 40°, 75°, 85°?
8. Prove, by a contrario reasoning, that if x is a non-constructible number, then also the following numbers are non-constructible: $\frac{1}{2}x$, $\sqrt{x}$, $\sqrt{x+1}$, x^2, x^2+2, x^2+3x+1. Can one say the same about number x^3?