Geometric constructions and elements of Galois' theory
List 4. Algebraic numbers, their minimal polynomials, and degrees.

Warm-up exercises.

1. Find polynomials with integer coefficients, whose roots are, subsequently, the following numbers:

$$\sqrt[4]{5},\ \frac{1}{\sqrt{2}},\ \sqrt{2}+\sqrt{3},\ \sqrt{1+\sqrt{3}},\ \frac{\sqrt{5}}{\sqrt[3]{4}},\ \frac{1}{\sqrt{2}}+\sqrt{2}.$$

2. Show that each algebraic number of degree 2 is a constructible number.
3. Divide (perhaps getting a remainder) polynomial x^3-x^2+3x-4 by the polynomial x^2+2x+2.
4. Check that number 1 is a root of the polynomial x^3-2x^2+1, and express this polynomial in the form $(x-1)\cdot Q(x)$, with explicit $Q(x)$.

Exercises.

1. Find polynomials with integer coefficients, whose roots are, subsequently, the following numbers::

$$1-\sqrt{2}+\sqrt{3},\ 1+\sqrt[3]{2},\ \sqrt[5]{3}-1,\ \sqrt{2}+\sqrt[3]{2},\ \frac{\sqrt{5}}{\sqrt[3]{2}-1}.$$

2. Prove that the following numbers are algebraic: $\cos 10^\circ$ oraz $\cos 18^\circ$.
   Hint: Use a trigonometric formula for cosinus of an appropriate multiplicity of a given angle (in our case, $3\times 10^\circ$ or $5\times 18^\circ$); you can derive this formula e.g. by taking the real part of the de Moivre formula.
3. Find a polynomial of degree 2, with integer coefficients, having the number $3-2\sqrt{2}$ as one of its roots. Verify that this polynomial is minimal for this number, so that this number is algebraic of degree 2.
4. Let $a=\sqrt[4]{q}$ be an irrational root of a rational number q. Can it be an algebraic number of degree 2?
5. Decide wheather the polynomial $2x^3-4x^2+3$ is irreducible over Q (i.e. wheather it can be decomposed into product of polynomials with rational coefficients).
6. Find a polynomial of degree 3, with integer coefficients, whose one of roots is the number $a=\sqrt[3]{2}-\sqrt[3]{4}$. Verify that this polynomial is minimal for this number. Conclude that a is an algebraic number of degree 3.
   HINT: calculate a^2 and a^3, and then write an equation

$$x_0+x_1\cdot a+x_2\cdot a^a=a^3$$

   with unknown coefficients x_0, x_1, x_3; solve this equation by transforming it into a system of 3 linear equations, where each equation corresponds to comparing coefficiets corresponding to $\sqrt[3]{2}$, $\sqrt[3]{4}$ and "free" coefficients. To verify minimality, check that you obtained polynomial is irreducible over rationals (in this degree this reduces to checking that it has no rational root).
7. Similarly as in exercise 6, find a polynomial of degree 4, with integer coefficients, having the number $b=\sqrt{2}+\sqrt{3}+\sqrt{6}$ as one of its roots. To do it, express b^4 as a linear combination of the numbers 1, b, b^2 i b^3.
8. Prove that if $W(x)$ is a minimal polynomial of an irrational algebraic number u of any degree, then W has no rational root.
9. Show that the algebraic number $\sqrt[4]{2}$ has degree 4. HINT: verify that the obvious polynomial of degree 4 having this number as its root cannot be expressed as product of two polynomials of degree 2 having rational coefficients, because this number cannot be a root of degree 2 polynomial with rational coefficients (the latter also requires an argument); complete further details of an argument for indecomposability of this polynomial over rationals, which will give its minimality.