Geometric constructions and elements of Galois' theory
List 5
Field extensions, their degrees; non-algebraic number π

Warm-up exercises.

1. Given two numbers $2-\sqrt[3]{2}+3\sqrt[3]{4}$ and $\sqrt[3]{4}-7\sqrt[3]{2}+5$ from the field $Q(\sqrt[3]{2})$, express their sum, difference, product and quotient in a form $q_0+q_1\sqrt[3]{2}+q_2(\sqrt[3]{2})^2$, where q_0, q_1, q_2 are rational.
2. Decide what is the degree and find any basis for the extension $Q \subset Q(\sqrt[5]{2})$, assuming we know that $\sqrt[5]{2}$ is an algebraic number of degree 5.
3. Let $F_1 = Q(\sqrt{3})$ and $F_2 = F_1(\sqrt{5})$. Find the degree and a basis for the extension $Q \subset F_2$.

Exercises.

1. Express the numbers $\frac{1}{\sqrt[3]{2}-1}$ and $\frac{1}{\sqrt[3]{2}-\sqrt[3]{4}+1}$ from the field $Q(\sqrt[3]{2})$ in the form
$$q_0+q_1\sqrt[3]{2}+q_2\sqrt[3]{4}, \quad \text{with } q_0, q_1, q_2 \in Q.$$
2. Let u be a root of the polynomial x^3-x^2-1, and let $a = 2+u-3u^2$, $b = 1-u+u^2$. Express the numbers $a\cdot b$ oraz $1/a$ in the form $q_0+q_1u+q_2u^2$, with q_0, q_1, q_2 rational.
3. Niech $F_1 = Q(\sqrt{2})$, $F_2 = F_1(\sqrt{3})$ i $F_3 = F_2(\sqrt{\sqrt{2}+\sqrt{3}})$. Znajdź stopień i bazę rozszerzenia $Q \subset F_3$.
4. Verify that the extension $Q(1+\sqrt{5})$ of the field Q of rationals coincides with the extension $Q(\sqrt{5})$.
5. Prove that each extension of degree 2 of any number field F is a quadratic extension. HINT: consider any basis for this extension of the form $\{1, a\}$; express a^2 as a linear combination of elements of this basis (with non-explicit coefficients from F); use this to show that a is a root of some quadratic equation with coefficients in F; express a using the well known formula for roots of a quadratic equation; make final conclusions.
6. Show that if u is an algebraic number of degree k, and if a is a distinct from u root of the minimal polynomial of u, then a is also an algebraic number of degree k.
7. Using the fact that the number $\sqrt[3]{2}$ is not constructible, show that the number $2\sqrt[3]{4}+\sqrt[3]{2}+1$ is also not constructible. Show also (without finding the minimal polynomial of b) that b is of degree 3 (the same degree as $\sqrt[3]{2}$). HINT: for non-constructibility of b you can use two arguments; first, assume a contrario that b IS constructible, and observe that then $\sqrt[3]{2}$ is a root of a quadratic equation $2x^2+x+1=b$ with constructible coefficients, so it is given be a formula for solutions of this equation; second argument is more abstact, and uses the fact that $b \in Q(\sqrt[3]{2})$ and b is not rational, so we have $Q \subset Q(b) \subset Q(\sqrt[3]{2})$, and we can deduce the degree of the extension $Q \subset Q(b)$.

Exercise based on the fact that π is a non-algebraic number.

8. Is it possible, by using compass and straightedge, to construct a circle whose area equals:
   (a) the area of a given square (inverse of the quadrature of a circle); HINT: assume that the side of the given square is a unit;
   (b) half of the area of a given circle;
   (c) sum of the areas of two given circles;
   (d) the difference af aread of a given square and the circle inscribed in this square?
9. Is it possible, by using compass and straightedge, to construct:
   (a) the circle whose perimeter is equal to a given segment;
   (b) the circle whose perimeter equals the sum of perimeters of three given circles;
   (c) the semicircle whose perimeter is equal to a given segment;
   (d) the semicircle whose perimeter is equal to the perimeter of a given circle?