Geometric constructions and elements of Galois' theory
List 6
Applications of Eisenstein's criterion

Warm-up exercises.

1. For which of the following polynomials one can deduce their irreducubility over the rational numbers by direct application of Eisenstein's criterion:
$x^4 + 6x^2 - 18x + 12, \qquad x^4 + 21x^3 - 3x^2 + 49, \qquad x^5 - 10x^2 + 50?$

Exercises.

1. Find the minimal polynomial of the number $\sqrt[3]{2+\sqrt{6}}$. Deduce that this number is not constructible.
2. Find the degree of the algebraic number $\sqrt{3-\sqrt[4]{3}}$.
3. For which natural numbers k and n can one deduce, by direct application of Eienstein's criterion, that the number $\sqrt[n]{k}$ is an algebraic number of degree n?
4. Show that if a polynomial $W(x) \in Q[x]$ is irreducible over Q, and if q is a rational number distinct from zero, then the polynomial $V(x) := W(x+q)$ is also irreducible over Q.
5. (a) Show that $\sqrt[5]{36}$ is an algebraic number of degree 5. HINT: consider the polynomial $V(x) := W(x+1)$, where $W(x) = x^5 - 36$.
   (b) Using a similar method, show that the degrre of the number $\sqrt[5]{4}$ is 5, and then deduce that this number is not constructible.
6. Prove that if u is an algebraic number, and if $q \in Q$, then the degree of the number $u+q$ is equal to the degree of u. HINT: consider the minimal polynomial $W(x)$ of the number u, and the polynomial $V(x) := W(x-q)$.
7. Given a polynomial $W(x) \in Q[x]$ which is irreducible over Q, and a rational non-zero number q, show that the new polynomial $V(x)$ dscribed as $V(x) := W(q \cdot x)$ is also irreducuble over Q. Deduce from this, that for any algebraic nymber u the degrees of the numbers u and $q \cdot u$ are equal.
8. Show that the polynomial $W(x) = 2x^4 - 7$ is irreducible by considering the associated polynomial $V(x) := W(\frac{x}{2})$ multiplied by such a factor, that its coefficients become integer. Likewise, show that the polynomial $9x^5 + 5$ is irreducible over Q.
9. Determine the degree of the number $\sqrt[5]{3/2}$ in the following two ways:
   (a) using the fact that $\sqrt[5]{3/2} = \frac{1}{2}\sqrt[5]{48}$ and applying exercise 7;
   (b) using the mothod from exercise 8.
   Similarly, determine the degree of the number $\sqrt[4]{2/3}$.
10. For a given polynomial $W(x) = a_n x^n + \ldots + a_1 x + a_0$ define its *palindrome* $\widetilde{W}(x)$ as
$$\widetilde{W}(x) := a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n.$$
   (a) Verify that $\widetilde{W}(x) = x^{\mathrm{st}(W)} \cdot W(\frac{1}{x}) = x^n \cdot W(\frac{1}{x})$.
   (b) Show that if a number $u \neq 0$ is a rooth of W, then the number $\frac{1}{u}$ is a root of $\widetilde{W}$.
   (c) Justify that if $W(x) = U(x) \cdot V(x)$, then $\widetilde{W}(x) = \widetilde{U}(x) \cdot \widetilde{V}(x)$.
   (d) Prove that if $W \in Q[x]$ is irreducible over Q then $\widetilde{W}$ is also irreducible.
   (e) Deduce that for any algebraic number $u \neq 0$ its inverse $\frac{1}{u}$ is also algebraic and its degree coincides with the degree of u.
   (f) Determine the degree of the number $\sqrt[5]{9/5}$.
11. Knowing that $W(x) = x^3 - 15x^2 + 12$ is the minimal polynomial of an algebraic number u, find miniamal polynomials of the numbers $u-2$, $3u$, $2u+3$, $\frac{1}{u}$, $\frac{1}{u+1}$, $\frac{2}{2u-1}$, $\frac{u-1}{u+1}$.