Exercises - Combinatorial Group Theory
List 1. Free Group.

Consequences of the definition

1. Let G be a free group with respect to S, and let $T \subset S$. Show that the subgroup $H = \langle T \rangle$ generated by T is free with respect to T.
2. Let G be any group with a generating set S. Show that G is (isomorphic to) a quotient of the free group F_S. Deduce that each group is a quotient of some free group.
3. Show that the free group of any rank greater than 1 is nonabelian.
4. (a) Show that any free group P has the following universality property (called *projectiveness*). For any groups G, H any *surjective* homomorphism $\gamma : G \to H$, and any homomorphism $\pi : P \to H$, there is a homomorphism $\phi : P \to G$ such that $\gamma\phi = \pi$.

   (b) Show that any projective group P is isomorphic with a subgroup of some fgree group.

   (Since we will see soon that any subgroup of a free group is free, it follows from (a) and (b) that free groups are characterized by the property of projectiveness.)
5. Show that if $G \to F$ is a homomorphism onto a free group F, and if N is the kernel of this homomorphism, then G is isomorphic with some semi-direct product of N by F. Show that if F is not free, then this is not necessarily the case.

Consequences of the construction (description) of free groups

6. Prove that each free group is torsion-free (i.e. contains no nontrivial element of finite order).
7. Prove that any free group of rank ≥ 2 has trivial center.
8. For any element $a \in G$, let $i_a : G \to G$ be the inner automorphism given by $i_a(g) = aga^{-1}$. Prove that if F is a free group of rank ≥ 2, then for distinct elements $a \in F$ the automorphisms i_a are distinct. Show thta the map $a \to i_a$ is a homomorphism. (In this way, the group F canonically embeds in its automorphism group $\mathrm{Aut}(F)$.)
9. Show that two nontrivial elements of a free group commute if and only if they are both powers of some third element.

   Hint: (1) first show that if these commuting elements are represented as reduced words u, w, then for some word x (possibly empty) we have $u = x\bar{u}x^{-1}$, $w = x\bar{w}x^{-1}$, where $\bar{u}, \bar{w}$ represent elemnts that also commute, and the word $\bar{u}\bar{w}$ is eithr reduced, or in its reduction process all of $\bar{u}$ or all of $\bar{w}$ is annihilated; (2) for commuting elements represented by words such as $\bar{u}$ and $\bar{w}$ above, apply induction with respect to the sum of lengths $|\bar{u}| + |\bar{w}|$.
10. Show that all abelian subgroups of free groups are cyclic.
11. Show that th subgroup $H = \langle Q \rangle$ in a group which is free with respect to $S = \{a, b\}$ generated by the set $Q = \{a^{-n}ba^n : n \geq 1\}$ is free with respect to Q. Deduce that the free group F_2 contains subgroups isomorphic with F_k for each $k \geq 1$.

Commutator subgroup and abelianization

Recall that for any elements a, b of a group G their *commutator* is the element $aba^{-1}b^{-1}$ (denoted $[a, b]$). *Commutator subgroup* of a group G is the subgroup generated by all commutators, i.e. the subgroup $[G, G] = \{[a, b] : a, b \in G\}$.

12. Show that the commutator subgroup of any group is its normal subgroup. Hint: show first that conjugation of any commutator is also a commutator (of some other elements).
13. Prove that the quotient group $G/[G, G]$ is always abelian. More generally, if $[G, G] < N \triangleleft G$ then G/N is abelian.

The group $G/[G, G]$ is called the *abelianization* of G, and it is denoted G^{ab}.

14. Prove that the abelianization of the free group F_S is isomorphic to the group Z^S, i.e. the direct sum of $|S|$ copies of the infinite cyclic group Z (or equivalently, the group of all functions $S \to Z$ with finite support, where we multiplu the functions pointwise). Hint: Consider the natural homomorphism $F_S \to Z^S$ and show that its kernel coincides with the commutator subgroup of F_S.

Conjugacy classes in free groups

A *cyclic transposition* of a word w is a word of form vu for any expresion $w = uv$ of w as concatenation of two subwords.

15. Show that in a free group F_S a word w' obtained as a cyclic trasposition of a word w represents an element which is a conjugate in F_S of the element represented by w.

Two words are *cyclically equivalent* if one of them can be obtained from the other by a finite sequence consisting of elementary operations and cyclic transpositions. A word is *cyclically reduced* if it is reduced and its last letter is not the inverse of the first letter.

16. Prove that:
    (a) any two cyclically equivalent words represent conjugate elements;
    (b) each equivalence class of the cyclic equivalence relation contains precisely one (up to cyclic transposition) cyclically reduced word;
    (c) two words represent some conjugate elements of the group F_S if and only if their cyclic reductions coincide (up to cyclic transposition).

    Note that part (c) is the solution of the *conjugacy problem* in free groups (i.e. it provides an algorithm for deciding whether two elements expressed in terms of generators are conjugate).

Other

17. Show, by referring to appropriate properties of free groups (either mentioned before, or specially derived for this occasion) that the following groups are not free: $SL(2, Z)$, Z^n for $n > 1$, the additive group Q of rationals, the direct sum (cartesian product) of any two nontrivial groups.