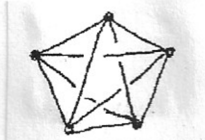
Combinatorial Group Theory. Exercises.
List 6.

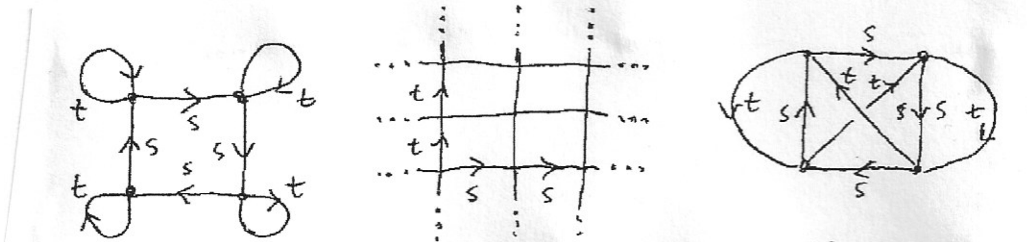
0. Show that if a finite connected graph is planar (i.e. embeds in the plane) then its connectivity rank is equal to the number of bounded componebts in the complement of its any embedding in the plane.

1. Indicate few distinct maximal trees and calculate rank of the fundamental group of the following graph:



2. Let X be a finite graph, and let T_1, T_2 be some two maximal trees in X. Let S_1, S_2 be the two bases of the fundamental group $\pi(X, v)$ induces by the trees T_1, T_2, respectively. We interprete the identical automorphism of the group $\pi(X, v)$ as an isomorphism $h : F_{S_1} \to F_{S_2}$. Describe h in the case when T_1 and T_2 differ by precisely one edge.

3. Choose some maximal trees in the following graphs



(a) (b) (c)

and use them to describe free generating sets (in terms of appropriate loop paths in the graphs) for the subgroups of the free group $\langle s, t \,|\, - \rangle$ induced by these graphs viewed as coverings of the graph

$X =$ 

4. Let X be the graph from the previous exercise. Describe the coverings of X corresponding to (a) the subgroup $\langle s \rangle$, (b) kernel of the homomorphism onto an infinite cyclic group $\langle a \,|\, - \rangle$ induced by $s \to a, t \to 0$.

5. Use coverings to determine all index 2 subgroups of a rank 3 free group F_3, and describe those subgroups in terms of some freely generating sets.

6. Use coverings to show that the subgroup in the free group $\langle s, t \,|\, - \rangle$ generated by s^2t^3, s^3t^2 has rank 2 and infinite index as a subgroup. Show that the subgroup (of the same group) generated by s, t^2, tst^{-1} has rank 3 and finite index (find this index).

7. Prove that any finitely generated normal subgroup of a free group F_k has finite index. Hint: use the symmetry properties of coverings corresponding to normal subgroups.

8. Prove that the fundamental group $\pi(X, v)$ of a connected graph X does not change under the following modifications of X:
   (a) subdivision of an arbitrary edge into two edges;
   (b) adding to X one new vertex, and one new edge connecting this new vertex with some "old" vertex.

9. Let $f_i : (Y_i, v_i) \to (X, v)$, for $i = 1, 2$, be some two connected coverings of a connected graph X, and suppose that they induce the same subgroup $H < \pi(X, v)$. Show that the graphs with base vertices (Y_i, v_i) are *isomorphic as coverings* of X, which means that there is an isomorphism $j : (Y_1, v_1) \to (Y_2, v_2)$ which commutes with f_1, f_2 (i.e. satisfies $f_2 j = f_1$).
10. Let $f : (Y, v') \to (X, v)$ be a covering map of connected graphs, and let $G = \pi(X, v)$, $H = f_*(\pi(Y, v')) < G$.
    (A) For any $u \in f^{-1}(v)$ let γ_u be some pathg in Y from v' to u, and let $g_u := [f_\# \gamma_u] \in G$. Verify that $\{g_u : u \in f^{-1}(v)\}$ is a set of representatives of right cosets of H in G. (This gives a direct proof of the fact that the index $[G : H]$ is equal to the multiplicity of the covering f.)
    (B) Denote by $Aut_f Y$ the group of *covering automorphisms* of the covering f, i.e. the group
    $$\{\phi \in Aut(Y) : \phi f = f\}.$$
    Show that $Aut_f Y \cong N_G(H)/H$, where $N_G(H) = \{g \in G : gHg^{-1} = H\}$ is the so called *normalizer* of the subgroup H in G.
11. Prove that for any positive integer j a finitely generated group G has finitely many subgroups of index j. Hint: reduce this exercise to the case when G is a free group (of finite rank), and then use coverings.
12. Show that for any $j > 2$ the free group F_2 contains a subgroup of index j which is not normal.
13. Find a subgroup of index 6 in the free group F_2, such that its index in its normalizer is 2.
14. Let X_S be the standard graph with one vertex for which $\pi(X) = F_S$, and let (Y, v) be the covering of X_S corresponding to a subgroup $H < F_S$. Let T be a maximal tree in Y such that for any vertex $u \in V_Y$ we have the equality of the distances $d_T(u, v) = d_Y(u, v)$.
    (a) Show that a maximal tree T as above always exists.
    (b) Prove that the free basis of the subgroups H induced in the standard way by a maximal tree T as above is reduced in the sense of Nielsen.

    [Observe that this proves the existence of a Nielsen-reduced basis for any subgroup H, including infinitely generated ones (while our previous arguments worked exclusively for finitely generated ones).]