Foundation of geometry and non-euclidean geometry
List 1. Axioms of area for polygons.

A *polygon* is any figure in the plane which can be expressed as the union of finitely many pairwise non-overlapping triangles. (Two figures are *non-overlapping* if they have no common interior points or, equivalently, if their intersection is empty or contained in the boundary of each of them.)

Axioms of area for polygons:

- (unit axiom - AJ)
  for a fixed unit square K_0 in the plane its area equals 1, i.e. $P(K_0) = 1$;
- (additivity axiom - AS)
  if $F = F_1 \cup F_2$ is the union of two non-overlapping polygons F_1 and F_2 then $P(F) = P(F_1) + P(F_2)$;
- (congruence axiom - AP)
  if two polygons F_1 and F_2 are congruent then their areas are equal, i.e. $P(F_1) = P(F_2)$;
- (monotonicity axiom - AM)
  if F_1, F_2 are polygons such that $F_1 \subset F_2$ then $P(F_1) \leq P(F_2)$.

Exercises:

1. Using axioms of area and the properties previously obtained from these axioms (during one of the lectures), and using commonly known in geometry properties of figures, derive in a precise deductive way formulas for area of: (a) a parallelogram, (b) an arbitrary triangle, (c) a trapezoid. If you have trouble with the general case, consider first easy special cases for these figures: not much oblique parallelogram, right triangle, isosceles triangle or trapezoid. In later parts of this exercise you can also refer to its earlier parts.
2. Verify which of the axioms of area are satisfied, and which are not, by the functions P_i described below:
   (a) $P_1(W) := 1$ for each polygon W;
   (b) $P_2(W) := 1/4 \cdot O(W)$, where $O(W)$ is the perimeter of W;
   (c) let C be a fixed circle on the plane, and let $P_3(W)$ be the length of this part of C which is contained in W; in particular, if $C \cap W = \emptyset$ then we have $P_3(W) = 0$, and if $C \cap W$ consists of more than one arc, we take as $P_3(W)$ the sum of lengths of these arcs; if the answer depends on the choice of C, e.g. on mutual position of C and K_0, discuss relevant cases of such a choice choice;
   (d) consider it grid points on the plane, i.e. points with both coordinates (with respect to some fixerd in advance cartesian coordinate system) integer; put $P_4(W) := w + 1/2 \cdot b - 1$, whre w ios the number of grid points in the interior of W, and b is the number of grid points on the boundary of W;
   (e*) denote by $conv(W)$ the so called *convex hull* of W, i.e. the smallest convex figure containing W; note that if W is a polygon, then so is $conv(W)$; put $P_5(W) := 1/4 \cdot O(conv(W))$, where O denotes the perimeter.
3. Prove that if we replace in the set of four axioms of area one of the axioms with a new axiom, as described in any of the items below, then we get an equivalent system of axioms:

(a) unit axiom AJ replaced with an axiom: "area of a fixed in advance square K_2 of side lengths 2 is equal to 4", i.e. "$P(K_2) = 4$";
(b) monotonicity axiom AM with an axiom: "$P(W) \geq 0$ for any polygon W";
(c) congruence axiom AP with the following weaker axiom: "congruent **triangles** have equal areas";
(d*) unit axiom with an axiom: "area of some fixed equilateral triangle T_0 with sides of length 1 is equal to $\sqrt{3}/4$ ".

4. Show that the following statements cannot be derived deductively from the axioms of area without referring to the congruence axiom AP (i.e. solely on the base of the three remaining axioms):
   (a) each unit square has length 1;
   (b) area of any triangle obtained by splitting a rectangle along its diagonal equals half of the area of cthis rectangle;
   (c) there are polygons of arbitrarily large areas;
   (d) any two parallelograms with the same base and the same height have equal areas.
5. Show that the statements (b), (c) and (d) of the exercise 4 cannot be derived without referring to the additivity axiom. Can one prove, without referring to additivity axiom, that any rectangle of both side lengths ≥ 1 has area ≥ 1?
6. Let P_0 denote an ordinary traditional area function for polygons. Check which of the axioms of area are satisfied, and which are not, for the functions P_i described below:
   (a) $P_6(W) := 1/2 \cdot P_0(W) + 1/2$;
   (b) $P_7(W) := \min(P_0(W), 1000)$;
   (c) fix some half-plane E containing the unit square K_0 (the square appearing in the unit axiom AJ), and let $P_8(W) := P_0(W \cap E)$.
7. Given the set of 4 axioms of area, replace the additivity axiom AS with the fiollowing new axiuom: "$P(W_1 \cup W_2) \leq P(W_1) + P(W_2)$ for any two polygons W_1, W_2". Is the new set of axioms equivalent to the original one?
8. Consider the following statemant: "among all rectangles with one side length fixed, anyone with the other side longer has greater area". Is this statement independent of the set of axioms:
   (a) AS, AM, AJ; (b) AP, AM, AJ?
9. Can one confute (i.e. disprove) the statement "area of any polygon equals half of square of its diameter" (where *diameter* denotes the maximal distance between two points of the polygon)
   (a) by referring to all axioms of area;
   (b) by referring to the axioms AP, AM and AJ only?
10. Is the set of axioms consisting of the axioms AP, AM, AJ and the negation of the axiom AS consistent (i.e. contradiction free)?
11. Is the set of axioms consisting of the axioms AS, AP, AJ and the axiom "areas of similar polygons are equal" (where *similarity* means the same shape but not necassarily the same size) consistent (i.e. contradiction free)?
12. For any set of axioms from exercises 10 or 11 which is consistent, decide wheather it is also complete.