Foundation of geometry and non-euclidean geometry
List 4. Half-plane model of non-euclidean geometry.

Recall that by our convention the half-plane model is given in coordinates as $\{(x,y) : y > 0\}$.

Warm-up exercises - to do yourself before the class

1. Find the measures of both angles between the following pairs of lines (by calculating first their cosines, and then using cosine tables, or a calculator with inverse trigonometric functions, or some other equivalent tools)
   (a) $x^2+y^2=3$, $(x-1)^2+y^2=4$;
   (b) $x^2+y^2=5$, $x=-1$.
2. Determine the equation of the non-euclidean line (i.e. of the appriopriate euclidean semi-circle) which passes through the points $(-1,2)$ and $(2,1)$.
3. Determine the equation of the non-euclidean line perpendicular to the line $L : x^2+y^2=25$ and passing through
   (a) the point $(3,4)$ lying at L;
   (b) the point $(1,7)$.
4. Calculate the equations of both non-euclidean lines which intersect the line $x=2$ at the point $(2,5)$ at an angle of measure $\pi/3$.
5. Deterine the equations of both non-euclidean lines passing through the point $(1,1)$ and asymptotic to the line $(x-2)^2+y^2=20$.
6. Show that for any two non-asymptotic non-euclidean half-lines there is always precisely one line asymptotic to both of them.

Exercises

1. Determine the line L which is perpendicular to the line $(x-7)^2+y^2=4$ and which intersects the line $x=6$ at an angle $\pi/6$.
2. Calculate (explicitely in degrees, with chosen approximation) the sum of angle measures in a non-euclidean triangle with the following vertices:
   (a) $(0,1)$, $(1,1)$ i $(3,1)$;
   (b) $(0,1)$, $(0,2)$ i $(1,2)$.
3. Determine the equations of both bisectors of the angles between the lines $x^2+y^2=4$ and $x=-1$.
4. Indicate (precisely) the following objects in the half-plane model:
   (a) a triangle with angle sum equal to $3\pi/4$;
   (b) and obtuse angle whose domain is contained in the domain of an acute angle;
   (c) a perpendicular to ane arm of an acute angle which does not intersect the other arm of this angle;
   (d) a 4-gon with three right angles, and the fourth angle of measure $\pi/3$.
5. Prove directly that any line asymptotic to the line $x=4$ has no common perpendicular with this line.
6. Find the common perpendiculars for the following pairs of divergent lines:
   (a) $x^2+y^2=1$ and $x^2+y^2=4$; (b) $x^2+y^2=1$ and $x=2$;
   (c) $x^2+y^2=1$ and $(x-1)^2+y^2=10$; (d) $x^2+y^2=1$ and $(x-4)^2+y^2=4$.
7. Construct an example of an ideal triangle with one ideal vertex and with angles $\pi/3$ at the remaining two vertices.
8. Given an ideal 4-gon (with all four ideal vertices), its "diagonals" are the lines which "connect" pairs of its opposite ideal vertices. For any positive $\alpha \leq \pi/2$ construct an example of such an ideal 4-gon whose "diagonals" intersect at angle α.
9. Show that
   (a) a horocycle and a line, (b) two distinct horocycles,
   may have at most 2 common points.
10. Prove that a line intersecting a horocycle at two distinct points forms equal angles of intersection at these points.
11. Show that, given any two points in the non-euclidean plane, there are precisely two horocycles passing through both of these points. Consider appropriate cases of mutual position of the given two points in the half-plane model.