Foundation of geometry and non-euclidean geometry
List 2. Axioms and models of eucliedean geometry of the plane.

1. For each of the below mentioned models of the theory of euclidean plane

(i) complete precise interpretations of all basic notions of of this theory,
(ii) give examples of all objects appearing in the axioms, defined in terms of basic notions,
(iii) verify which of the axioms are satisfied in these models, and which are not.

   A. The role of points is played by real numbers, and there is just one line, and all points belong to this line.
   B. The role of points is played by the letters of English alphabet, and the role of lines is played by words from some favorite dioctionary.
   C. The role of points is played by points from the interior of some fixed circle, and the role of lines is played by (interiors of) the chords of this circle; the relation of order on the lines, the measure of segments, and the measure of angles are taken as those same concepts in ordinary geometry.
   D. The role of the plane is played by the 3-dimensional space, and the other notions are interpreted naturally, and coincide with the corresponding notions in ordinary geometry.
   E. All the basic notions are the same as in ordinary geometry, except the measure of angles, which is taken to be half of the ordinary measure.
   F. All the basic notions are the same as in ordinary geometry, except the measure of segments, which are taken to be squares of the ordinary measure.
   H. The role of points is played by points of some unit semi-sphere without the boundary equator, and the role of lines is played by the parts of the great circles contained in this semi-sphere. The order relation is interpreted naturally, measures of segmants are taken as arc lengths of arcs representing segments. Measure of angles (between half-lines represented by certain arcs in great circles) are taken as angle measures between lines tangent to these arcs at the point representing the angle vertex.

Incidence theory is the theory which is a small part of the euclidean geometry of the plane, and which deals only with points, lines and the relation of incidence as the basic notions. This theory is based on the axioms of incidence I1-I3 and on the parallel axiom R.

2. In each of the drawings below, the "thickened dots" represent points, and curve segments (straight or curved) represent lines. The incidence relation corresponds to the fact that a curve segment passes through a thickened dot. These pictures give us models of the incidence theory. Verify for which of these models all the axioms of incidence theory are satisfied.
3. For each of the axioms of incidence theory decide wheather it is independent from the remaining axioms.
4. Can one deduce from the axioms of incidence theory that
   a. each line contains at least 3 points?
   b. the plane consists of at least 4 distinct points?
   c. there is at least 6 distinct lines in the plane?
   d. at least 3 lines pas through each point?
   e. for each line there are at least 2 lines which do not intersect this line?

   Which of the above statements (a)-(e) are independent from (the axioms of) the incidence theory?
5. Is the incidence theory consistintent? Is it complete? Justify your answers.

In the remaining exercises provide the proofs based solely on the axioms, and on previous exercices (or previous points of the exercises).

6. Lines p_1, p_2 are called *parallel* either if they coincide or if they do not intersect (i.e. do not contain any common point, or equivalently, do not pass through a common point). Show that:
   a. the relation of being parallel is transitive;
   b. if $p \neq q$ and p intersects q, then each line parallel to p intersects q at precisely one point; mopreover, for any point of q there is precisely one line parallel to p and passing through this point.
7. We call a *direction* any set of all lines parallel to a given line. Show that:
   (a) for any point and any direction, there is precisely one line in this direction which passe through this point;
   (b) there are at least three distinct directions;
   (c) distinct directions are disjoint as sets of lines (i.e. they cannot contain a common line);
   (d) each direction consists of the same number of lines;
   (e) lines from the same direction consist of the same number of points.
8. Show that if p and q are distinct lines, then there is line r which intersects both p and q at a single point.
9. Show that any two lines contain the same number of points.
10. We call a *pencil* any set of all lines passing through a given point. Show that any two pencils have the same cardinality.
11. Show that the number of points in each pencil is greater by 1 than the number of points in each line.