

Exercises - Algebraic Topology 1. List 1

We assume that all considered topological spaces are pathwise connected, and all considered maps are continuous.

1. Verify that the operation of composition for paths satisfies the following cancellation condition: if $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1$, then $f_0 \simeq f_1$.
2. Show directly from the definition that for any two loops f, g in X based at $x_0 \in X$ we have the following: $f \sim g \Leftrightarrow f \cdot \bar{g} \sim \text{const}_{x_0}$, where const_{x_0} is the constant loop based at x_0 , and where \bar{g} is the inverse loop of g given by $\bar{g}(t) = g(1 - t)$.
3. Justify that for any topological space X the following three conditions are equivalent:
 - (a) each map $S^1 \rightarrow X$ is homotopic to a constant map;
 - (b) each map $S^1 \rightarrow X$ extends to a map $D^2 \rightarrow X$, where D^2 is the 2-dimensional disk, whose boundary coincides with the considered S^1 ;
 - (c) $\pi_1(X, x_0) = 0$ for any $x_0 \in X$.

Deduce from the above that a space X is simply connected (i.e. $\Pi_1 X = 0$) if and only if any two maps $S^1 \rightarrow X$ are homotopic.

4. Show that if $\pi_1 X = 0$ (i.e. if the fundamental group of X is trivial) then any two paths in X connecting any pair of points $x_0, x_1 \in X$ are homotopic.
5. We say that a topological space X (not necessarily pathwise connected) is *contractible*, if there is a map $F : X \times I \rightarrow X$ such that $F(x, 0) = x$ oraz $F(x, 1) = x_0$ for any x and for some fixed x_0 . Show that if X is contractible, then it is pathwise connected, and $\pi_1 X = 0$ (in other words X is then simply connected).
6. Prove that any convex subset of R^n is contractible.
7. Let T be a finite *tree*, i.e. a connected finite graph with no closed cycles made of edges. Show that $\pi_1 T = 0$.
8. Verify that the homomorphism $\varphi_d : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ (corresponding to a change of a base point) depends only on the homotopy class of the path d from x_0 to x_1 .
9. Let G be a *topological group*, i.e. a group equipped with topology, for which the maps $m : G \times G \rightarrow G$ and $r : G \rightarrow G$ given by $m(g, h) := g \cdot h$ and $r(g) := g^{-1}$ are continuous. Show that $\pi_1(G, e)$ is then a commutative group.

A free homotopy between loops f and g in X (based not necessarily at the same point) is a family of loops $f_t : t \in I$ in X depending continuously on t (which means that the map $(s, t) \rightarrow f_t(s)$ is continuous), such that $f_0 = f$ and $f_1 = g$, and the base points of the loops f_t may vary with the parameter t .

10. Show that if any loop in X is freely homotopic to some constant loop, then $\pi_1 X = 0$. BEWARE: in general, freely homotopic loops based at the same point need not be homotopic (compare the next exercise).
11. Let $[S^1, X]$ be the set of equivalence classes of the relation of free homotopy for loops in X (based at all points). Let $\Phi : \pi_1(X, x_0) \rightarrow [S^1, X]$ be the natural assignment provided by the fact that any homotopy of loops is an example of their free homotopy. Prove that
 - (a) Φ is a surjection;
 - (b) $\Phi([f]) = \Phi([g])$ if and only if the elements $[f]$ and $[g]$ are conjugate in the group $\pi_1(X, x_0)$ (i.e. there is $h \in \pi_1(X, x_0)$ such that $[g] = h^{-1}[f]h$).

Given a subspace $A \subset X$, a **retraction of X onto A** is a continuous map $r : X \rightarrow A$ for which $r|_A = \text{id}_A$ (i.e. $r(x) = x$ for each $x \in A$).

12. Let $A \subset X$ be a *retract*, i.e. a subspace for which there is a retraction $R : X \rightarrow A$. Show that for any $x_0 \in A$ the natural map $\pi_1(A, x_0) \rightarrow \pi_1(X, x_0)$ induced by the inclusion map $A \rightarrow X$ is injective.
13. Deduce from the previous exercise that a space with nontrivial fundamental group cannot be a retract of a simply connected space.

A **deformation retraction** is a retraction $r : X \rightarrow A$ for which there is a homotopy $r_t : X \rightarrow X, t \in I$ (continuous as the map $X \times I \rightarrow X$) such that: (1) $r_0 = \text{id}_X$, (2) $r_1 = r$, (3) for each $t \in I$ we have $r_t|_A = \text{id}_A$. A subspace A as above is then called a **deformation retract** of X .

14. Show that if $r : X \rightarrow A$ is a deformation retraction, then $r_* : \pi_1 X \rightarrow \pi_1 A$ is an isomorphism.
15. Deduce from the previous exercise that if $\pi_1(Y, y_0) \neq 0$, then $X \times \{y_0\}$ is not a deformation retract of the product $X \times Y$.
16. It follows from the isomorphism $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$ that any two loops from the subspaces $X \times \{y_0\}$ and $\{x_0\} \times Y$ (both based at (x_0, y_0)) represent commuting elements in the fundamental group $\pi_1(X \times Y, (x_0, y_0))$. Describe explicitly a homotopy of loops which justifies this fact.
17. Let A be a path connected subspace of X containing the base point x_0 . Show that the homomorphism $i_* : \pi_1(A, x_0) \rightarrow \pi_1(X, x_0)$ induced by the inclusion $i : A \hookrightarrow X$ is surjective if and only if each path in X with both endpoints in A is homotopic to some path in A .