Exercises - Algebraic Topology 1.   List 2

1. Show that any homomorphism $\pi_1 S^1 \to \pi_1 S^1$ can be realized as the induced homomorphism φ_* for some map $\varphi : S^1 \to S^1$.
2. Consider the map $f : S^1 \times I \to S^1 \times I$ given by $f(e^{i\theta}, s) = (e^{i(\theta+2\pi s)}, s)$ (so that at the boundary circles) $S^1 \times \{0\}$ and $S^1 \times \{1\}$ it coincides with the identity. Prove that f is homotopic to the identity through a homotopy f_t equal to the indentity on the circle $S^1 \times \{0\}$ for all t, but it is **not** homotopic to the identity through a homotopy f_t coinciding for all t with identities on both boundary circles. Hint: see what f does with the points $(1, s) : s \in I$.
3. Show that there is no retraction $r : X \to A$ when:
   (a) $X = R^3$ and A is its any subspace homeomorphic to S^1;
   (b) $X = S^1 \times D^2$ is a solid torus and $A = S^1 \times S^1$ is its boundary torus;
   (c) $X = S^1 \times D^2$, and A is the embedded circle as on the picturte below;

A

   (d) X is the union of two disks D^2 glued along single boundary points in each of them, and A is the corresponding union of the boundary circles of these two disks.
4. Show that the spaces in the following pairs are not homeomorphic:
   (a) S^2 and D^2; (b) S^2 and S^n for $n \neq 2$.
5. Let X be the space obtained from the disk D^2 after gluing to each other some two distict boundary points of this disk.
   (a) Prove that $\pi_1 X = Z$.
   (b) Is the subspace $A \subset X$ obtained from the boundary of D^2 (and homeomorphic to two circles glued together along single points in them) a retract of X?
6. Is the boundary of the Möbius band a retract of the whole band?
7. Show that any path connectyed open neighbourhood U of a point x in the plane, after deleting this point, has a nontrivial fundamental group.
8. Use the technical fact presented during the lecture which allowed to show homotopy invariance of the fundamental group (i.e. Lemma 1.19 in the book of Hatcher) to prove the following. Let $f_t : X \to X$ be a homotopy for which the maps f_0 and f_1 are the identities. Then for any $x_0 \in X$ the loop $f_t(x_0)$ represents an element from the center of the fundamental group $\pi_1(X, x_0)$.
9. Let A be a retract of a pathwise connected space X, and suppose that $\pi_1 A$ is a normal subgroup in $\pi_1 X$. Show that then $\pi_1 X = \pi_1 A \times [\pi_1 X / \pi_1 A]$.
10. Prove directly, without referring to Seifert-van Kampen theorem, that if X is the union of its two open and simply connected subspaces, $X = U \cup V$, for which the intersection $U \cap V$ is pathwise connected, then $\pi_1 X = 0$.

11. Use the previous exercise to get an alternative proof of the fact that the spheres S^n for $n \geq 2$ are simply connected.
12. Justify that, for $n \geq 3$ and for any finite set P of points in R^n, the space $R^n \setminus P$ is simply connected. Prove the same for S^n substituted in place of R^n.
13. Let X be a finite union of lines in R^n passing through the origin $0 \in R^n$. Show that for $n \geq 4$ we have $\pi_1(R^n \setminus X) = 0$.

Exercises concerning the notion of homotopy equivalence

Exercises 3-4 and 9-12 on pages 18-19 of the Hatcher's book "Algebraic Topology" (exercises at the end of Chapter 0), and the exercises below.

Given a continuous map $f : X \to Y$, consider the space called rozważmy przestrzeń zwaną *the cylinder of* f, denoted M_f, and described as the quotient space of the disjoint union $(X \times [0,1]) \sqcup Y$ through the relation induced by the identifications of the form $(x,1) \sim f(x) : x \in X$ (with the quotient topology). Consider also *the cone of the map* f, denoted C_f, to be the quotient $C_f := M_f/(X \times \{0\})$, where $X \times \{0\}$ is viewed as the subset of M_f.

14. Show that the space Y viewed naturally as a subspace in the cylinder M_f (for any $f : X \to Y$) is a deformation retract of this cylinder. Why a similar argument does not work for $Y \subset C_f$?
15. Use the fact that $\pi_1(S^1) \neq 0$ for showing that, in general, the cone C_f (for $f : X \to Y$) is not homotopy equivelent to the space Y.
16. Verify that if $f : X \to Y$ is a homotopy equivelence, then the map $h : X \to M_f$ given by $h(x) = (x, 0) \in X \times \{0\} \subset M_f$ is also a homotopy equivalence.
17. Show that if the maps $f, g : X \to Y$ are homotopic then the cones C_f and C_g are homotopy equivalent.
18. Prove by referring directly to the definition that any finite graph X is homotopy equivalent with the wedge of $1 - \chi(X)$ copies of the circle, where $\chi(X)$ is the Euler characteristic of the graph X. HINT: to warm up, show first that a graph having the shape of the greek letter θ is homotopy equivalent to the wedge of two circles.