Exercises - Algebraic Topology 1. List 3

Basics of combinatorial group theory

Free products and group presentations

1. Show that the free product $G * H$ of non-trivial groups G and H has trivial center, and that it always contains elements of infinite order (even if the factors are finite).
2. Verify that
   (a) $\langle S_1|R_1\rangle * \langle S_2|R_2\rangle = \langle S_1 \sqcup S_2|R_1 \sqcup R_2\rangle$;
   (b) $\langle S_1|R_1\rangle \oplus \langle S_2|R_2\rangle = \langle S_1 \sqcup S_2|R_1 \sqcup R_2 \cup \{[s_1, s_2] : s_1 \in S_1, s_2 \in S_2\}\rangle$.
   Extend (or generalize) these observations to the case of the free product and the direct product of the arbitrary cardinality collections of factor groups.
   Deduce that the group Z^n has the presentation $\langle s_1, \dots, s_n|[s_i, s_j] : 1 \le i < j \le n\rangle$.
3. Show that the cyclic group Z_n has the presentation $\langle a|a^n\rangle$, and that the symmetric group S_3 has the presentation $\langle a, b|a^2, b^2, (ab)^3\rangle$.
4. Let $G = \langle S|R\rangle$, let ρ be an element of G expressed in terms of the generators from S, and let N be the normal subgroup of G induced (or generated as a normal subgroup) by ρ. Show that $G/N \cong \langle S|R \cup \{\rho\}\rangle$.

Commutator subgroup and abelianization

Recall that for any two elements a, b of a group G their *commutator* is the element $aba^{-1}b^{-1}$ (denoted as $[a, b]$). The *commutator subgroup* of G is the subgroup generated by all commutators, i.e. the subgroup $[G, G] = \{[a, b] : a, b \in G\}$.

5. Show that the commutator subgroup of any group is its normal subgroup. Hint: verify first that a conjugate of any commutator is also a commutator (of some other elements, in general).
6. Prove that the quotient group of form $G/[G, G]$ is always ablelian. More generally, if $[G, G] < N \triangleleft G$ then G/N is abelian.

The group $G/[G, G]$ is called the *abelianization* of G, and we denote it by G^{ab} or $\mathrm{Ab}(G)$.

7. Show that the abelianization of the free group F_S is isomorphic with the group Z^S, i.e. with the direct sum of $|S|$ copies of the group Z (or in yet another (though equivalent) terms, with the group of all functions $S \to Z$ having finite support, where multiplication is given as pointwise mutiplication). Hint: consider the natural homomorphism $F_S \to Z^S$, and prove that its kernel coincides with the commutator subgroup of the group F_S.
8. Show that $\mathrm{Ab}(G * H) = \mathrm{Ab}(G) \times \mathrm{Ab}(H)$, and more generally that $\mathrm{Ab}(*_\alpha G_\alpha) = \oplus_\alpha \mathrm{Ab}(G_\alpha)$.
9. Verify that the abelianization of the group given by the presentation $\langle S|R\rangle$ has the presentation
$$\langle S|R \cup \{[s, s'] : s, s' \in S\}\rangle.$$