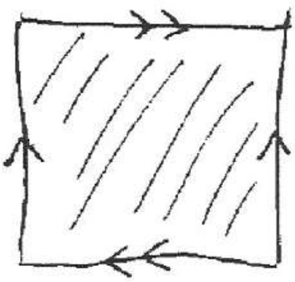
Exercises - Algebraic Topology 1. List 4

Applications of van Kampen theorem, and more...

1. Let X be a finite connected graph.
   (a) Show that $\pi_1 X$ is a free group F_n, for some n. Hint: Consider any maximal tree T inside the graph X, and view X as the union of T and of the cycles C in X, each of which contains a single edge of $X \setminus T$ (and its remaining part is contained in T); more precisely, consider small open neighbourhoods in X of T and of all cycles C as above. Yoy may also use induction with respect to the number of edges outside a maximal tree.
   (b) Show that if X is contained (embedded) in the plane, then n coincides with the number of bounded components of the complement $R^2 \setminus X$.
   (c) Coming back to arbitrary graphs X (i.e. not necessarily planar) prove that the number n from part (a) depends only on the Euler characteristic $\chi(X)$ of the graph X, and derive the corresponding expression of n in terms of $\chi(X)$.
2. Let X be the space obtained from the sphere S^2 after idetifying the north pole N with the south pole S. Detremine the group $\pi_1 X$, either applying van Kampen theorem, or expressing X as a 2-dimensional cell complex, e.g. the presentation complex for some presentation.
3. Let Y be the space obtained from a pathwise connected space X by adding to it an n-dimensional cell, for some $n \geq 3$. Prove that the inclusion map $X \to Y$ induces an isomorphism of the corresponding fundamental groups (in particular, the fundamental group remains unchanged). Prove the same for an operation of adding simultaneosly an arbitrary family of n-dimensional cells.

The *Klein bottle* K is the surface obtained by gluing to each other the sides of a square accordingly with the picture below.



4. Verify that the fundamantal group $\pi_1 K$ of the Klein bottle has the presentation $\langle a, b | aba^{-1}b \rangle$.
5. Let $G = \langle a, b | aba^{-1}b \rangle$ be the fundamental group of the Klein bottle.
   (a) Using the homomorphism $G \to Z$ induced by the assignments $a \to 1$ and $b \to 0$, show that the element corresponding to the generator a has infinite order in G.
   (b) Prove that the subgroups $\langle a \rangle$ and $\langle b \rangle$ in G, generated by the elements a and b, respectively, are both normal.
   (c) Check that if we assign to the element a the real function $f(x) = -x$, and to the element b the function $g(x) = x + 1$, then these assignments induce (extend to) a homomorphism of the group G to the group of bijections of the set of real numbers.

(d) Use the homomorphism from the previous point (c) to show that the element b in the group G has infinite order.

(e) Prove that the group G is non-abelian.

(f) Justify that the circle corresponding to the loop b in the Klein bottle K is not a retract of K. Use Exercise 9 from List 2, and some previous parts of this exercise.

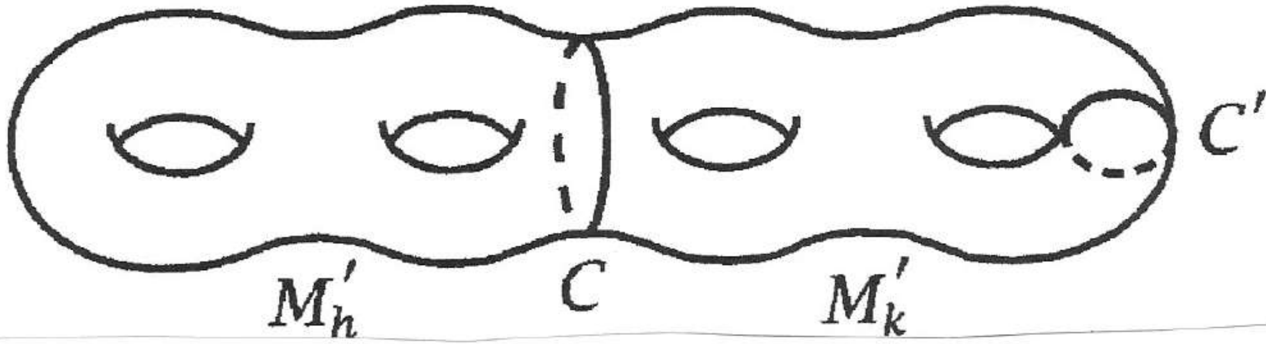
6. Prove using algebraic arguments that groups given by the presentations $\langle a, b | aba^{-1}b \rangle$ and $\langle c, d | c^2 d^2 \rangle$ are isomorphic. Show that the second group is the fundamental group of the space Y obtained by gluing two copies of the Möbius band by a homeomorphism of their boundaries. Show also that Y is actually homeomorphic to the Klein bottle, and deduce the above isomorphism of groups topologically.

7. (a) Let X be the space obtained from the torus T by deleting the interior of some small disk $D \subset T$. Show that there is no retraction of X onto the boundary circle $\partial X = \partial D$.

   (b) Prove a similar result for an orientable surface M_g of arbitrary genus $g > 1$.

8. (a) Let C be a closed curve which split the surface M_g into two components homeomorphic to the surfaces M_h and M_k with deleted interiors of disks, where $h \geq 1$ and $k \geq 1$ (see the figure below). Check that there is no retraction of M_g onto C.

   (b) Let C' be a nonseparating closed curve on the surface M_g surrounding one of the handles of this surface, as on the figure below. Show that C' is a retract of M_g.



9. Justify that, by identifying all three boundary components of the disk with two holes via certain homeomorphisms, then we can obtain two non-homeomorphic topological spaces. Use abelianizations of the fundamental groups to distiguish these spaces.

10. Consider the arcs α and β in the cylinder $D^2 \times I$, as in the picture below. The curve γ is obviously contractible in this cylinder, but the intuition tells it is not contractible in the complement of the union $\alpha \cup \beta$. Verify this fact.