Algebraic Topology 2. Exercises.
List 1.

0. Let $\sigma, \sigma' : \Delta^n \to X$ be any two maps whose restrictions to the boundary of Δ^n coincide. Show that $\sigma - \sigma'$ is then an n-cycle in X.
1. Let $s : [0,1] \to X$ be any path in a topological space X. Consider the following two singular 1-simplices $\sigma_i : \Delta^1 \to X$, $i = 1, 2$: $\sigma_1((1-t)e_0 + te_1) = s(t)$ and $\sigma_2((1-t)e_0 + te_1) = s(1-t)$.
   (1) Prove that $\sigma_1 + \sigma_2$ is a 1-cycle.
   (2) Prove that $\sigma_1 + \sigma_2$ is null-homologous, by describing an explicit 2-chain $a \in C_2X$ with $\sigma_1 + \sigma_2 = \partial a$.
2. A singular 1-simplex $\sigma : \Delta^1 \to X$ is called a *loop* if $\sigma(e_0) = \sigma(e_1)$.
   (a) Show that each loop is a 1-cycle.

   Two loops σ_0, σ_1 are *freely homotopic* if there is a continuous map $F : \Delta^1 \times [0,1]$ such that
   - for each $x \in \Delta^1$ we have $F(x,0) = \sigma_0(x)$ and $F(x,1) = \sigma_1(x)$,
   - for each $t \in [0,1]$ the map $\sigma_t : \Delta^1 \to X$ given by $\sigma_t(x) := F(x,t)$ is a loop.

   (b) Prove that any two freely homotopic loops are homologous, i.e. they induce the same element in the homology group H_1X.

   A 1-chain $\sigma_0 + \ldots + \sigma_{r-1}$ such that $\sigma_i(e_0) = \sigma_{i-1}(e_1)$ for each $i \in Z/rZ$ is called an *elementary 1-cycle.*

   (c) Prove that each elementary 1-cycle is a 1-cycle.
   (d) Prove that each elementary 1-cycle is homologous with some loop.
   (e) Show that the elements in the homology group H_1X induced by loops generate this group.
   (f) Prove that if X is path-wise connected then each element in the homology group H_1X is induced by a loop.
   (g) Prove that if X is path-wise connected, and if $\pi_1X = 0$, then $H_1X = 0$.
3. Prove that the homomorphisms $H_kX \to H_kY$, for $k > 0$, induced by the maps $f : X \to Y$ which are constant, are trivial.
4. Let $A \subset X$ be a retract, and let $r : X \to A$ be a retraction map, i.e. a continuous map such that $r(x) = x$ for all $x \in A$. Denote also by $i : A \to X$ the corresponding inclusioon map. For each integer $k \geq 0$, denote by $r_k : H_kX \to H_kA$ and $i_k : H_kA \to H_kX$ the homomorphisms induced by r and i, respectively.
   (1) Show that each i_k is injective.
   (2) Show that for each $k \geq 0$ we have $H_kX \cong H_kA \oplus \ker(r_k)$.
5. Verify that for homotopic maps f, g the induced homomorphisms f_*, g_* of **reduced** homology groups coincide.
6. Verify that chain homotopy is an equivalence relation.
7. Check, both directly from the definition and by applying the exact sequence for pairs, what is the relationship between the homology groups H_nX and $H_n(X,x)$, where $x \in X$ is any point.

Exercises 15,16, 17(a), 20, 21, 27 and 29 from pages 132-133 of Hatcher's book "Algebraic Topology".