Algebraic Topology 2. Exercises.
List 2.

1. Let $\rho : \Delta^n \to D^n$ be a homeomorphism (which maps $\partial\Delta^n$ onto $\partial D^n = S^{n-1} \subset D^n$). Further, let $\tau : \partial\Delta^{n+1} \to S^n$ be a homeomorphism, and view it as a chain in $C_n S^n$ (taking formally $\tau = \sum_{i=0}^{n+1}(-1)^i \tau|_{[e_0,\dots,\hat{e}_i,\dots,e_{n+1}]}$). Given some basepoint $x_0 \in S^n$, let $\nu : \partial\Delta^{n+1} \to S^n$ be a continuous map with the following properties:
   - ν maps the interior of the face $[e_1,\dots,e_{n+1}]$ of Δ^{n+1} homeomorphically onto $S^n \setminus \{x_0\}$, and it maps the boundary of this face onto x_0;
   - ν maps all other faces of Δ^{n+1} onto x_0.

   View ν as a chain in $C_n S^n$, by taking $\nu = \sum_{i=0}^{n+1}(-1)^i \nu|_{[e_0,\dots,\hat{e}_i,\dots,e_{n+1}]}$ Finally, let $\mu : \Delta^n \to S^n$ be a continuous map which sends the interior of Δ^n homeomorphically onto $S^n \setminus \{x_0\}$, and which maps the boundary of Δ^n onto x_0; view μ as a chain in $C_n S^n$.
   (a) Check that ρ is a relative cycle in (D^n, S^{n-1}), and show that it induces a generator in the homology group $H_n(D^n, S^{n-1}) \cong Z$.
   (b) Check that τ is a cycle in $C_n S^n$, and show that it induces a generator in the homology group $H_n(S^n) \cong Z$.
   (c) Check that, for $n \geq 1$, ν is a relative cycle in $(S^n, \{x_0\})$, and show that it induces a generator in the homology group $H_n(S^n, \{x_0\}) \cong Z$.
   (d) Check that, for $n \geq 1$, μ is a relative chain in $(S^n, \{x_0\})$, and show that it induces a generator in the homology group $H_n(S^n, \{x_0\}) \cong Z$.

   HINTS: the assertions of (a)-(d) should be proved simultaneously, using induction over the dimension n, by the arguments similar to those used to calculate $H_n S^n$ and $H_n(D^n, S^{n-1})$. Use also, without proof, the intuitive fact that any map τ is homotopic to some map ν as above, and vice versa.
2. Let $r : S^n \to S^n$ be a reflection with respect to some equatorial $S^{n-1} \subset S^n$, and let H^n_+, H^n_- be the hemi-spheres of S^n bounded by this S^{n-1}. Let $\sigma : \Delta^n \to H^n_+$ be a homeomorphism (which sends $\partial\Delta^n$ onto $S^{n-1} = \partial H^n_+$). Check that, for $n \geq 1$, the chain $c = \sigma - (r \circ \sigma) \in C_n S^n$ is a cycle, and show that it induces a generator in the homology group $H_n S^n \cong Z$.

   HINT: let x_0 be the pole of S^n contained in the interior of H^n_-, and let $h : S^n \to S^n$ be some map, homotopic to the identity, which sends the interior of H^n_+ homeomorphically onto $S^n \setminus \{x_0\}$, and sends all of H^n_- onto x_0 (such a map clearly exists); consider then the cycle $h_\#(c)$ homologous to c, and compare it with the cycle ν of exercise 1(c); finally, use the assertion of exercise 1(c).
3. By using a local homology argument, show that given a finite graph Γ (viewed as a topological space), there is no homeomorphism of Γ that sends a vertex of Γ to a vertex with different degree. (Clearly, there are easier arguments to show this fact, but we want to practice local homology.)

Exercises 1-4 and 7-8 from page 155 of Hatcher's book "Algebraic Topology".