Algebraic Topology 2. Exercises.
List 4.

CW-complexes, cellular homology, simplicial homology

1. Verify that for any CW-complex X the pair (X^n, X^{n-1}) is a good pair of spaces.
2. Provide details of the (inductive) argument for showing that any compact suibset of a CW-complex X is contained ion some finite subcomplex of X.
3. Check that if the image of the characteristic map φ_α for a cell e^{n+1}_α is disjoint with the (interior of) a cell e^n_β then the incidence coefficient vanishes, i.e. $i_{\alpha,\beta} = 0$.
4. For a given finite CW-complex X the cellular boundary map $\partial^{CW} : C^{CW}_{n+1}X \to C^{CW}_n X$ is the "linear" map given by the matrix with integer coefficients $i_{\alpha,\beta}$ (incidence coefficients for pairs of $(n+1)$- and n-cells). Check how is this matrix modified if one canges
   (a) the orientation of one of the $(n+1)$-cells,
   (b) the orientation of one of the n-cells.
5. Given a cellular map $f : (X, A) \to (Y, B)$ between CW-pairs, describe (in terms of the associated degrees $f_{\alpha,\beta}$) the induced chain homomorphism $f_\# : C^{CW}_\star(X, A) \to C^{CW}_\star(Y, B)$.
6. Recall that we have identifications $C^{CW}_n X = H_n(X^n, X^{n-1})$, and that under these identifications the cellular boundary map $\partial^{CW}_{n+1} : C^{CW}_{n+1}X \to C^{CW}_n X$ is given as the composition $j_n\partial_{n+1}$ of the maps $\partial_{n+1} : H_{n+1}(X^{n+1}, X^n) \to H_n X^n$ and $j_n : H_n X^n \to H_n(X^n, X^{n-1})$.
   (a) Show that vthe map ∂^{CW}_{n+1}, viewed as a homomorphism $H_{n+1}(X^{n+1}, X^n) \to H_n(X^n, X^{n-1})$, coincides with the boundary map in the long exact sequence of the triple (X^{n+1}, X^n, X^{n-1}).
   (b) Using part (a) and naturality of exact sequences of triples, show that for any cellular map $f : X \to Y$ the cellular induced homomorphisms $f^{CW}_\# : C^{CW}_n X \to C^{CW}_n Y$ commute with the cellular boundary homomorphisms ∂^{CW} (i.e. they form a morphism of cellular chain complexes).
7. Consider a closed connected n-dimensional manifold M with a fixed triangulation. Suppose that this manifold is orientable, i.e. the n-simplices of its triangulation can be oriented *consistently*, which means that for any $(n-1)$-simplex τ of this triangulation the orientations induced from the orientations of the two n-simplices containing τ are opposite.
   (a) Using simplicial homology, show that $H_n M = Z$.
   (b) Suppose M is closed, connected, n-dimensional, triangulated and non-orientable. Show that then $H_n M = 0$.
8. Let $K(3, 3, 3)$ be a 2-dimensional simplicial complex described as follows. Consider sets A, B, C consisting of 3 elements. Identify the vertex set of $K(3, 3, 3)$ with the disjoint union $A \sqcup B \sqcup C$, and the set od 2-simplices with the family of all such subsets $T \subset A \sqcup B \sqcup C$ which have precisely one element in each of A, B and C. Compute the simplicial homology of $K(3, 3, 3)$.

Cellular homology and Euler characteristic

Exercises 15–16 and 20–24 from pages 156–156 of Hatcher's book "Algebraic Topology".

Mayer-Vietoris sequences

Exercises 28–29 and 31–33 from pages 157–158 of Hatcher's book "Algebraic Topology".

9. Give an elementary derivation for the Mayer-Vietoris sequence in simplicial homology for a simplicial complex X decomposed as the union of subcomplexes A and B.