DIFFERENTIAL TOPOLOGY - EXERCISES
LIST 1. Measure zero sets in manifolds and Sard's Theorem

1. Show that for any space curve $\gamma : (a, b) \to R^3$ of class C^2 which is *regular* (i.e. $\forall t \in (a, b)$ $\gamma'(t) \neq 0$) there is a plane $P \subset R^3$ such that the orthogonal projection of γ on P is still regular.
2. Show that if $m < p$ then any C^1 map of a manifold M^m to the sphere S^p is homotopic to a constant map (i.e. it can be continuously deformed into a constant map). HINT: show that any map as above omits some point $x \in S^p$, and use stereographic projection $\sigma : S^p \setminus \{x\} \to R^p$.
3. Find a funcyion $f : R \to R$ such that its set of critical values is Q (rationals).
4. Deduce from Sard's Theorem that if $f : M \to N$ is smooth and $\dim M < \dim N$ then $f(M)$ has measure 0.
5. Consider an n-dimensional manifold X, a natural number $m > 2n + 1$, and given a vector $v \in R^m \setminus \{0\}$ let π_v be the orthogonal projection in R^m onto a linear subspace (a hyperplane) $Ort(v) \subset R^m$ consisting of all vectors orthogonal to v.
   (A) Show that if $f : X \to R^m$ is a injective C^1 map then there is $v \in R^m \setminus \{0\}$ such that the composition map $\pi_v \circ f : X \to Ort(v)$ is still injective.
   Hint: consider an auxiliary map $h : \{(p, q) \in X \times X : p \neq q\} \to RP^{m-1}$ given by $h(p, q) = [f(p) - f(q)] \in RP^{m-1}$.
   (B) Show that if $f : X \to R^m$ is an immersion of class C^2 then there is $v \in R^m \setminus \{0\}$ such that the composition map $\pi_v \circ f : X \to Ort(v)$ is still an immersion.
   Hint: define and use an appropriate auxiliary map
   $$g : TX \setminus \{\text{zero tangent vectors in } TX\} \to RP^{m-1}$$
   induced by an immersion f.
6. It is known that each manifold can be smoothly embedded in R^N, for sufficiently large N.
   (A) Use this fact, and part (A) of the previous exercise, to show that each compact n-dimensional manifold can be smoothly embedded into R^{2n+1}.
   (B) Show that each n-dimensional manifold admits an immersion inoto R^{2n}.
7. A *contour* of the projection of a surface $P \subset R^3$ onto the plane $\{z = 0\}$ is the set of the images (through the above projection) of all those points $x \in P$ for which the tangent plane T_xP contains the vector $\frac{\partial}{\partial z}$. Show that the contour of any smooth surface $P \subset R^3$ is a zero measure subset of the plane $\{z = 0\}$.
8. Given a smooth curve $\gamma : R \to R^2$, consider a parametrized family of curves $\gamma_u : u \in R^2$ given by $\gamma_u(t) = \gamma(t) + u \cdot t$. Prove that arbitrarily close to $(0, 0)$ there exist $u \in R^2$ such that the curves γ_u are regular.
9. Given a smooth function $f : R \to R$, consider a parametrized family of functions $f_u : u = (u_1, u_2) \in R^2$ given by $f_u(t) = f(t) + u_1 \cdot t + u_2 \cdot t^2$. Show that arbitrarily close to $(0, 0)$ there is $u \in R^2$ such that f_u is a so called *Morse function*, i.e. a function such that for any t with $f_u'(t) = 0$ we have $f_u''(t) \neq 0$.
10. Show that there is no smooth function $f : R^n \to R^n$ such that the set $f^{-1}(a)$ is uncountable for each $a \in R^n$.
11. (A) Show that any closed set $A \subset R^n$ is the zero set of some smooth function $f : R^n \to R$.
   (B) Use (A) and Sard's Theorem to show that for any closed $A \subset R^n$ there exist open subsets $U_1 \supset U_2 \supset U_3 \supset \ldots$ such that for each $j \geq 1$ the boundary of U_j in R^n is a smooth $(n-1)$-submanifold and $A = \bigcap_{j=1}^{\infty} U_j$.