DIFFERENTIAL TOPOLOGY - EXERCISES
LIST 2. Manifolds of jets

Warm-up exercises.

1. Describe the space of 0-jets $J^0(X, Y)$ and the 0-jet extension $j^0 f$ of any smooth map $f : X \to Y$.
2. Describe, using natural parametrizations, the spaces of jets $J^k(R, R)$, and the k-jet extensions $j^k f$ for smooth functions $f : R \to R$. Do the same for $J^k(R^n, R)$.
3. Describe, using natural parametrizations, the spaces $J^1(R^n, R^m)$ and 1-jet extensions $j^1 f$ of the maps $f : R^n \to R^m$ of class C^1.
4. Justify that the notion of rank of a jet (defined as the rank of the corresponding representative mapping at the corresponding point), is well defined for (a) 1-jets, and (b) for k-jets, where $k \geq 1$ is arbitrary.
5. Define a natural mapping $\pi_{k,l} : J^k(X, Y) \to J^l(X, Y)$ for $k > l$ and show that it is well defined, smooth, that it is a submersion, and even more precisely a locally trivial fiberation. Identify, up to diffeomorphism, the preimages $\pi^{-1}_{k,l}(\sigma)$ (i.e. the fibres of the fibration $\pi_{k,l}$).
6. Prove that the set of 1-jets of maximal rank is an open subset of the manifold $J^1(X, Y)$. Is the same true for $J^k(X, Y)$ with arbitrary $k > 1$?
7. Given arbitrary manifolds X and Y and arbitrary $k \geq 1$, construct some natural smooth embedding of the product $X \times Y$ into the manifold of jets $J^k(X, Y)$. Verify that this is indeed an embedding.

Essential exercises.

8. Prove that for any natural $r \leq \min(n, m)$ the set of jets of rank r is a submanifold in (a) $J^1(X^n, Y^m)$, (b) $J^k(X^n, Y^m)$ for arbitrary k.
9. Consider an algebraic oparation (which we call “multiplication”) in the jet space $J^k(R^n, R^n)_{0,0}$ (i.e. jets of the maps $f : R^n \to R^n$ such that $f(0) = 0$) induced by the composition of the smooth mappings $(R^n, 0) \to (R^n, 0)$.
   (a) Show that the invertible elements in $J^1(R^n, R^n)_{0,0}$ with respect to this multiplication can be identified with the set of matrices in $GL(n, R)$.
   (b) Prove that for any k the invertible elements in $J^k(R^n, R^n)_{0,0}$ constitute a Lie group (i.e. a group which is also a manifold, so that the group operations of multiplication and taking the inverse are smooth).
10. How can one embed $J^k(X, Y)$ in $J^l(X, Y)$ for $1 \leq k < l$? Is there any canonical embedding?

Characterizations of k-tangency (alternative definitions of k-jets).

Let $\gamma_1, \gamma_2 : R \to X$ be smooth curves on a manifold X. We say that these curves are *functionally k-tangent at $t_0 \in R$* if for any smooth real function $h : X \to R$ the difference function $h \circ \gamma_1 - h \circ \gamma_2 : R \to R$ has all derivatives of orders $0 \leq i \leq k$ vanishing at t_0 (in particular, the value of this difference function at t_0 is 0).

11. Prove that the curves γ_1, γ_2 as above are k-tangent at t_0 if and only if they are functionally k-tangent at t_0.

12. Let $f, g : X \to Y$ be smooth mappings of manifolds. Prove that these mappings are k-tangent at a point $p \in X$ if and only if for any smooth curve $\gamma : R \to X$ such that $\gamma(0) = p$ the composition curves $f \circ \gamma$ and $g \circ \gamma$ in Y are functionally k-tangent at 0.

"Algebraic" characterization of k-tangency

For $x \in X$ let $C^\infty_x(X, R)$ denotes the algebra of germs at x of smooth real functions $X \to R$. Let $\mathcal{M}(X, x) \subset C^\infty_x(X, R)$ be the ideal of germs of these functions that vanish at x. Let $\mathcal{M}(X, x)^k$ be the algebraic k-th power of the ideal $\mathcal{M}(X, x)$, i.e. the smallest subalgebra that contains products of k elements from $\mathcal{M}(X, x)$.

13. Prove that smooth mappings $f, g : X \to Y$ are k-tangent at a point $x \in X$ if and only if for any germ $\varphi \in C^\infty_y(Y, R)$, where $y = f(x) = g(x) \in Y$, we have $\varphi \circ f - \varphi \circ g \in \mathcal{M}(X, x)^{k+1}$.