DIFFERENTIAL TOPOLOGY - EXERCISES
LIST 3. Whitney topology

Preliminary exercises.

1. Verify that the strong topology on the space $C(\mathcal{X}, \mathcal{Y})$ of continuous maps $\mathcal{X} \to \mathcal{Y}$ is indeed stronger (as the name suggests) than the compact-open topology.
2. Show that for any smooth manifolds X, Y and for $r < k$ the Whitney C^r-topology on $C^\infty(X, Y)$ is weaker (essentially!) than the Whitney C^k-topology.
3. Show that if $\mathcal{X}$ is a locally compact space, and if $\mathcal{Y}$ is a metric space, then the compact-open topology on $C(\mathcal{X}, \mathcal{Y})$ coincides with the topology of uniform convergence on compact sets (which means that the operation of closure in this topology can be characterized in terms of uniform convergence on compact sets).
4. Show that if $\mathcal{X}$ is a paracompact space, and if $\mathcal{Y}$ is a metric space, then the strong topology on the space $C(\mathcal{X}, \mathcal{Y})$ is equivalently given by the local bases of neighbourhoods consisting of sets of the form
$$\mathcal{U}(f, \varepsilon) = \{g \in C(\mathcal{X}, \mathcal{Y}) : \forall x \in \mathcal{X} \ d_{\mathcal{Y}}(f(x), g(x)) < \varepsilon(x)\},$$
where $d_{\mathcal{Y}}$ is the metric on $\mathcal{Y}$, and where $\varepsilon : \mathcal{X} \to R_+$ is an arbitrary continuous function.
5. Verify that the Whitney C^r-topology, for $r \leq \infty$, is Hausdorff.

Exercises.

6. Construct a complete metric on the space $C^\infty([0,1], R)$ compatible with the Whitney C^∞-topology (and earlier do the same for Whitney C^r-topologies for $r < \infty$).
7. Show that the map $C^\infty(X, R) \times R \to C^\infty(X, R)$ given by $(f, c) \to c \cdot f$ is continuous if and only if the manifold X is compact.
8. Prove that the subset of $C^\infty(X, Y)$ consisting of all proper maps (i.e. maps for which the preimage of any compact set is compact) is open in C^∞-topology. Prove also openness of the subsets consisting of all immersions, and of all submersions.
9. Show that for any smooth $f : X \to Y$ the induced map $f^* : C^\infty(Y, Z) \to C^\infty(X, Z)$ given by $f^*(g) = g \circ f$ is continuous with respect to the C^∞-topology if and only if f is proper. Deduce that if W is a submanifold of X then the map $C^\infty(X, Y) \to C^\infty(W, Y)$ given by the restriction $f \to f|_W$ is continuous iff W is closed in X.
10. Suppose that a sequence f_n is convergent to g in the C^r-topology on $C^r(X, Y)$. Prove that then there is a compact subset $K \subset X$ such that for all sufficiently large n we have $f_n(x) = g(x)$ for all $x \in X \setminus K$.
11. Suppose that $f_t \in C^\infty(X, Y), t \in [0,1]$ is a deformation with compact support K, i.e. $f_t(x) = f_0(x)$ for all $x \in X \setminus K$ and all $t \in [0,1]$, *smoothly depending on parameter* t, i.e. such that the map $(t, x) \to f_t(x)$, $[0,1] \times X \to Y$ is of C^∞ class. Show that the map $[0,1] \to C^\infty(X, Y)$ given by $t \mapsto f_t$ is then continuous with respect to the C^∞-topology.
12. Justify that the set of injective immersions in the space $C^r(R^1, R^2)$ is not open in Whitney C^r-topology.
13. Show that the groupo $\mathrm{Diff}^r(X)$ of C^r-diffeomorphisms of a manifold X, for $r \leq \infty$, is a topological group with respect to the Whitney C^r-topology.
14. Prove that for any smooth $f : Y \to Z$ the induced map $f_* : C^\infty(X, Y) \to C^\infty(X, Z)$ given by $f_*(g) = f \circ g$ is continuous with respect to C^∞-topologies.