DIFFERENTIAL TOPOLOGY - EXERCISES
LIST 4. Transversality

Preliminary exercises.

1. Let V_1, V_2 be some linear subspaces in a vector space R^n, and view them as submanifolds. Show that $V_1 \pitchfork V_2$ if and only if $V_1 + V_2 = R^n$.
2. Let $F : R^n \to R^m$ be a linear map, and let V be a linear subspace in R^m. Show that $F \pitchfork V$ if and only if $F(R^n) + V = R^m$.
3. Which of the following pairs of linear subspaces cross transversely:
   (1) the plane Oxy and the plane span$[(0, 1, 1), (1, -1, 0)]$ in R^3;
   (2) the plane span$[(1, 3, 0), (2, -1, 0)]$ and the axis Ox in R^3;
   (3) $R^n \times \{0\}$ and the diagonal $\Delta = \{(x, x) : x \in R^n\}$ in $R^n \times R^n$;
   (4) the diagonal Δ and *the "skew diagonal"* $\{(x, -x) : x \in R^n\}$ in $R^n \times R^n$;
   (5) the subspaces of symmetric and skew-symmetric matrices in the space of all matrices $M_{n\times n}(R)$;
   (6) the subspace of symmetric (respectively, skew-symmetric) matrices, and the subspace of matrices with vanishing trace in the space $M_{n\times n}(R)$ of all matrices?
4. Let $f : R \to R$ be a smooth real function, and let $W \subset J^1(R, R)$ be the submanifold consisting of all such jets $j^1g(x)$ for which the function g has vanishing first derivative at x, for all $x \in R$. Justify that the condition $j^1f \pitchfork W$ is equivalent to the fact that for each critical point x of f (i.e. each point at which the derivative of f vanishes) f has a non-vanishing second derivative.
5. Let W be a submanifold in $R^{n+k} = R^n \times R^k$. Show that for a dense set of points $x \in R^n$ we have $W \pitchfork (\{x\} \times R^k)$. State and verify a similar observation concerning a smooth map $f : X \to R^{n+k}$ from a manifold X.

Exercises.

6. Check that $y_0 \in Y$ is a regular value of a smooth map $f : X \to Y$ if and only if the graph of the map f, viewed as a submanifold in $X \times Y$, is transversal to the submanifold $X \times \{y_0\}$.
7. Justify the following generalization of the observation from the previous exercise: a smooth map $f : X \to Y$ is transversal to a submanifold $W \subset Y$ if and only if the graph of f is transversal in $X \times Y$ to the submanifold $X \times W$ (and moreover, if and only if $j^kf \pitchfork (p^Y_k)^{-1}(W)$, for any $k \geq 1$, where $p^Y_k : J^k(X, Y) \to Y$ is the natural projection).
8. We say that a fixed point p of a smooth map $f : X \to X$ is *non-degenerate* if the differential $df_p : T_pX \to T_pX$ has no fixed point other than 0 (i.e. no non-zero eigenvector).
   (A) Show that any non-degenerate fixed point is isolated in the set of all fixed points of a smooth map $f : X \to X$.
   (B) Express the fact of having only non-degenerate fixed points (by a smooth map $f : X \to X$) in terms of an appropriate transversality condition.
9. We say that smooth maps $f : X_1 \to Y$ and $g : X_2 \to Y$ such that $f(a) = p = g(b)$ are *transversal at* (a, b) if $df_a(T_aX_1) + dg_b(T_bX_2) = T_pY$. Show that $f \pitchfork g$ at (a, b) if and only if the map $f \times g : X_1 \times X_2 \to Y \times Y$, $f \times g(x_1, x_2) = (f(x_1), g(x_2))$, is transversal to the diagonal $\Delta Y \subset Y \times Y$, $\Delta Y = \{(y, y) \in Y \times Y : y \in Y\}$, at (a, b).
10. Recall that the curvature of a regular curve $\gamma : R \to R^3$ at a point t is non-zero if and only if the vecors of the first and the second derivatives of γ at t are linearly independent. Express the property that the curvature of a regular curve γ does not vanish (i.e. it is non-zero at every point t) as some transversality condition (or a combination of few transversality conditions) in appropriate manifolds of jets.