DIFFERENTIAL TOPOLOGY - EXERCISES
LIST 6. Morse functions

0. Construct a Morse function with precisely 4 critical points on manifolds $S^1 \times S^2$ and $S^2 \times S^2$. What are the indices of the critical poits of these functions?
1. [Reeb] Justify that an n-dimensional closed manifold which admits a Morse function with precisely two critical points is **homeomorphic** to the sphere S^n. BEWARE: Milnor has discovered examples of manifolds as above which are **not diffeomerphic** to S^n.
2. Let $f : R^n \to R$ be a smooth function.
   (a) Show that if p is a critical point of f then the Jacobian at p of the gradient function ∇f of f coincides with the Hessian of f at p.
   (b) Show that f is a Morse function if and only the zero vector $0 \in R^n$ is a regular value of the gradient function ∇f.

   Try to generalize the above two observations to the case of arbitrary smooth manifold equipped with a Riemannian metric.
3. Is it true that any Morse function on a closed manifold M can be realized as the height function for some embedding of M in some R^N?
4. Let M be a smooth $(n-1)$-dimensional oriented submanifold of R^n, and let $p \in M$. Let L be a line in R^n orthogonal to M at p, which we will view as a copy of the reals. Let $f : M \to L$ be the restriction to M of the orthogonal projection of R^n to L. Show that p is a nondegenerate critical point of f if and only if p is a regular point of the Gauss map $G : M \to S^{n-1}$.

   Hint: Recall that the Gauss map associates to any point x of M the unit vector orthogonal to M at x oriented consistently with the orientations of M and R^n. Express M, locally near p, as a graph of a function $R^{n-1} \to R$.
5. Let $U \subset R^n$ be an open subset, and let $f : U \to R$ be an arbitrary smooth function. Prove that the set of all these $a = (a_1, \ldots, a_n) \in R^n$ for which the modified function

$$f_a(x) = f(x) + \sum_{i=1}^{n} a_i x_i$$

   is not a Morse function has measure zero.
6. Let M be a smooth closed submanifold in R^n. Show that:
   (a) the set of all vestors $v \in R^n$ for which the function $f_v : M \to R$ given by $f_v(x) = \langle v, x\rangle$ is a Morse function is open and dense in R^n;
   (b) the set of points $u \in R^n$ such that the function $f_u : M \to R$ given by $f_u(x) = ||u - x||^2$ is a Morse function is open and dense in R^n.

   Will the assertions (a) and (b) still hold true if instead of being a Morse function we demand that f is a Mortse function with pairwise distinct critical values?
7. Find all closed surfaces which admit a Morse function with
   (a) precisely three
   (b) precisely four

   critical points with pairwise distinct values.
8. Using the fact that adding an r-handle we change the Euler characteristic by the value $(-1)^r$, justify the following two facts:
   (a) each Morse function on a closed orientable surface of genus g has at least $2g + 2$ critical points;
   (b) the Euler characteristic of any closed manifold of odd dimension is 0 (hint: consider two Morse functions f and $-f$ on M).