Joint Meeting of UMI-SIMAI-PTM

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New results in the topological classification of Gromov boundaries of hyperbolic groups

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INFINITE GROUPS

- finitely generated group *G*
- *S* – finite generating set

(any element *g* of *G* can be expressed as

$$g = s_1 s_2 \ldots s_n$$

with each s_i a generator from S or its inverse)

THEOREM [B.H. Neumann, 1937]

There are uncountably many pairwise non-isomorphic groups with 2 generators.

- finitely presented group $G = \langle S, R \rangle$
- S – finite
- R – finite set of relations

  (i.e. equations for generators and their inverses)

THEOREM [Adyan, Rabin, 1958]

The isomorphism problem for finitely presented groups is undecidable.

- study some reasonable classes of f.p. groups
- up to equivalences weaker than isomorphism

Geometric viewpoint [geometric group theory]

- infinite f.g. groups as geometric objects
- Cayley graph $\Gamma(G, S)$

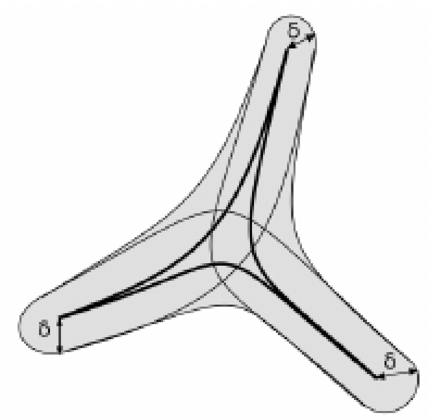
  vertices - $V_\Gamma = G = \{v_g : g \in G\}$

  unordered edges - $E_\Gamma = \{\{v_g, v_{gs}\} : g \in G, s \in S\}$

- connected, infinite, locally finite, regular graph
- G acts on $\Gamma(G, S)$ by automorphisms, transitively on vertices

Hyperbolic groups

f.g. group *G* is <u>hyperbolic</u> if for some (eq. for any) finite S there is δ>0 such that any geodesic triangle in *Γ(G, S)* is *δ*-thin.



<u>geodesic triangle</u> – 3 vertices of *Γ(G, S), and 3 connecting geodesics [a,b], [b,c], [c,a]*

<u>δ-thin</u> -
[a,b] is contained in δ-neighbourhood of [b,c] ᴗ [b,c], etc.

Numerous natural examples:
free groups, many lattices in Lie groups,
fundamnetal grooups of compact negatively curved spaces,
random groups, …...

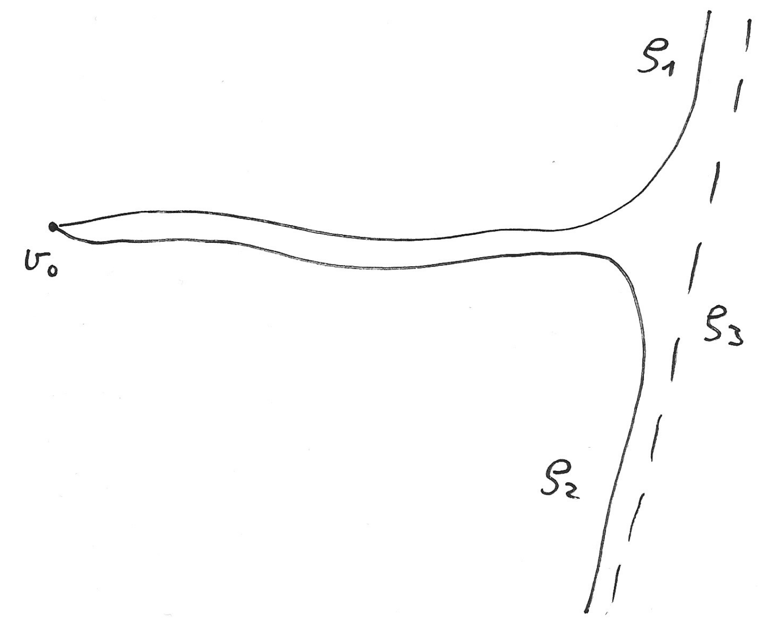
class is closed under: finite extensions,
free product (with amalgamation along finite subgroups)

Gromov boundary ∂*G*

∂*G* (as a set) = equivalence classes [ϱ]
of geodesic rays ϱ in *Γ(G, S)* based at fixed v_0
up to relation of ***fellow travelling***

topology on ∂*G*:

hyperbolicity → divergence of
non-asymptotic geodesic rays



<[ϱ1],[ϱ2]> := dist(v_0, ϱ_3)
(well defined up to 10δ)

the larger <[ϱ1],[ϱ2]> -
- the smaller distance between [ϱ1], [ϱ2] in ∂*G*

∂G – compact, metrisable, finite topological dimension

AIM: study Gromov boundaries of hyperbolic groups (up to homeomorphism)

- known spaces (up till recently) appearing as ∂G
  - n-spheres, $n>0$,
  - Menger universal compacta in dimensions $n = 1,2,3$
  - Sierpiński compacta in any dimension n
    ($n=0$ – Cantor set, $n=1$ – Sierpiński carpet)

- [Gromov]

  ∂G is disconnected iff $G = G_1 *_A G_2$, A – finite.

Turning to recent results/developments

- <u>Markov compactum</u> $\leftarrow$

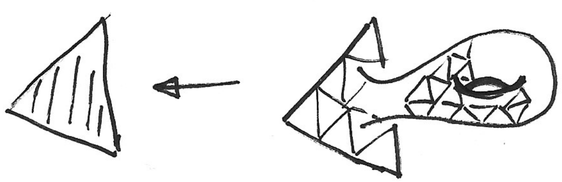
  a metrisable compact topological space *X* given by:

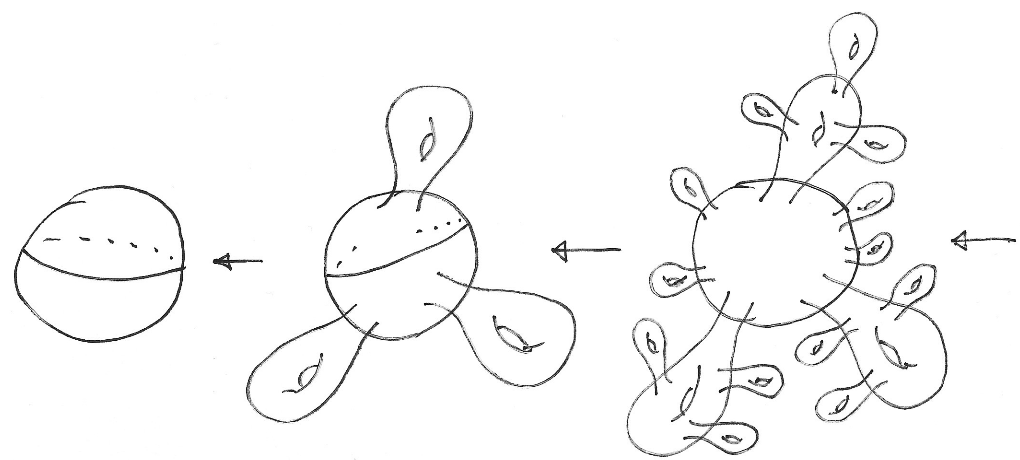
  - an algorithm based on finite data, yielding
  - $X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \ldots$ inverse sequence of finite simplicial complexes
  - $X := \textit{inv-lim}\ X_n$ (inverse limit)

- THEOREM [D. Pawlik, 2015]

  Gromov boundary of any hyperbolic group is (homeomorphic to) a Markov compactum.

- <u>trees of manifolds</u> – a subclass of Markov compacta
- An algorithm determined by:
  - a choice of a closed connected n-manifold M
  - *data:* X_0 = n-sphere (triangulated in a standard way),
    n-simplex $\leftarrow$ n-simplex # M (with appropriate triangulation)
- the resulting space depends uniquely on M (up to homeomorphism), has topological dimension n, is connected and homogeneous
  - we denote it $X(M)$





THEOREM [J. Świątkowski, 2016]
If $M = \partial W$ for some compact $(n+1)$-manifold W, then $X(M) \approx \partial G$ for some hyperbolic group G.

- <u>dense amalgam</u> A Ŭ B of compact metric spaces A and B =

  = a compact metric space X such that

  - connected components of X are:

    A_α – copies of A, B_β – copies of B, singletons

  - both the union of all A_α and the union of all B_β are dense in X

  - both families $\{A_\alpha\}$ and $\{B_\beta\}$ are null in X

  - separation by an open-dense set of any two connected components of X

THEOREM [J. Świątkowski, 2016]

(1) A Ŭ B is unique up to homeomorphism

(2) $\partial(G_1 *_A G_2) \approx \partial G_1$ Ŭ ∂G_2